Math 126 AA/AB

Worksheet 1

January 4, 2005

When finding the limit of a_n as $n \to \infty$, one way is to replace the expression for a_n with the function $f(x)$. It is important to note that n is an integer that equals $1, 2, 3, \ldots$ whereas x can equal any real number. Theorem 3 on page 704 of Stewart tells that if $lim_{x\to\infty} = L$ and $f(n) = a_n$ where n is an integer, then $lim_{n\to\infty} a_n = L$. *It is really important to understand the difference between* $lim_{n\to\infty} f(n)$ *and* $lim_{x\to\infty} f(x)$. *Carefully read pages 703 and 704.* If you have any questions, please email me.

Compute the limits as $n \to \infty$

1.a

$$a_n = \frac{3n^3 + 5}{n^2 + 7n - 1}$$

First rewrite $f(n) = a_n$ as $f(x)$, and then apply L'hopital's rule twice

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{3x^3 + 5}{x^2 + 7x - 1} \\
&= \lim_{x\to\infty} \frac{9x^2}{2x + 7} \\
&= \lim_{x\to\infty} \frac{18x}{2} \\
&= \infty
\end{aligned}$$

1.b

$$b_n = \frac{\ln(1 + 2e^n)}{n}$$

Like in the previous problem, rewrite $f(n) = b_n$ as $f(x)$, and then apply L'hopital's rule twice

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln(1 + 2e^x)}{x}$$

$$= \lim_{x\to\infty} \frac{\frac{2e^x}{1+2e^x}}{1} = \lim_{x\to\infty} \frac{2e^x}{1+2e^x}$$

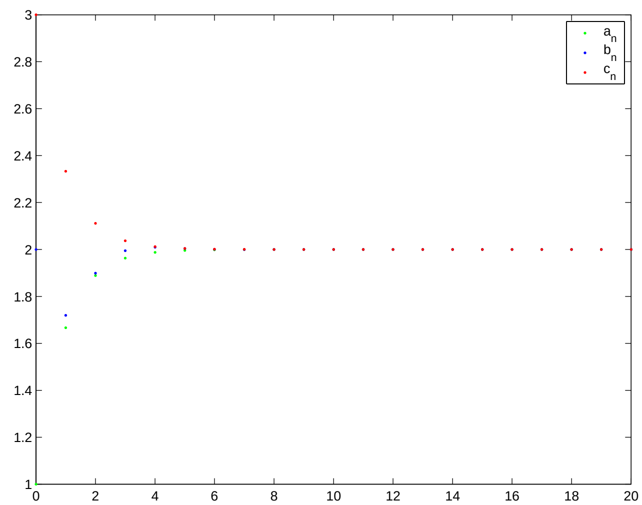
$$= \lim_{x\to\infty} \frac{2e^x}{2e^x}$$

$$= 1$$

1.c

$$b_n = 2 - \frac{\sin n}{3^n}$$

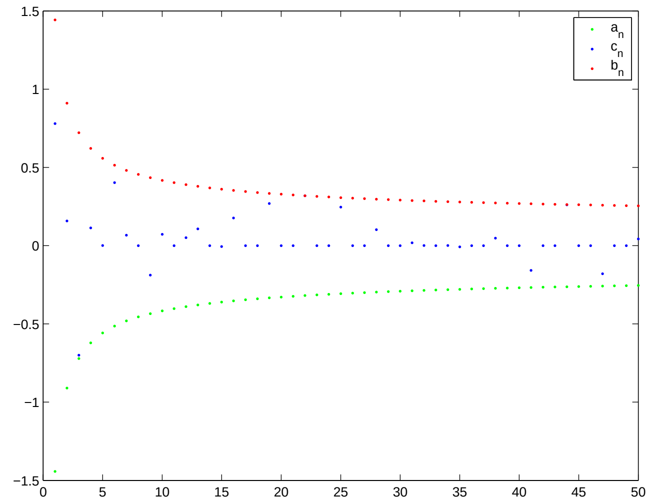
On this problem, we can use the squeeze theorem (page 705 of Stewart). If be let $a_n = 2 - \frac{1}{3^n}$ and $c_n = 2 + \frac{1}{3^n}$, then $a_n \le b_n \le c_n$. Also note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 2$. Therefore by the squeeze theorem, $\lim_{n\to\infty} b_n = 2$



1.d

$$c_n = \frac{(\cos n)^n}{\ln(n+1)}$$

This is similar to the previous problem. If we let $a_n = \frac{-1}{\ln(n+1)}$ and $b_n = \frac{1}{\ln(n+1)}$, then $a_n \le c_n \le b_n$. Also note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Therefore by the squeeze theorem, $\lim_{n\to\infty} c_n = 0$.



Bounded and Monotone Sequences

2.c

Showing that the sequence $a_n = f(n)$ is monotone can be shown by proving that f' doesn't switch signs. Thus, by using the quotient rule,

$$f'(x) = \frac{2x}{(x^2+1)^2} \geq 0 \qquad \forall x \geq 1$$

Because we have shown that f' doesn't switch signs for the region that we are interested in ($n = 1, 2, \ldots$ corresponds to $x \geq 1$), then the sequence is monotone. Furthermore, because $f' \geq 0$, the sequence is increasing, so the lower bound would be at $n = \frac{1}{2}$ and the upper bound would be the limit as $n \to \infty$.