AMath 301
Due Date: Early Tuesday morning, May 22nd

Homework 4

1. On the last homework, we used MATLAB's built in ODE solvers and the TOL setting to find the order of the particular ODE solver. Here we will code two different ODE solvers (Forward Euler, and Heun's method) to solve a particular differential equation. We will also find the order of these methods. Consider the ordinary differential equation:

$$y' = -3y\sin(t), \qquad y(0) = \frac{1}{2}.$$

which has an exact solution $y(t) = \frac{1}{2}e^{3(\cos(t)-1)}$, (Check this).

(a) Solve this ODE numerically using Forward Euler's method:

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

with $t = [0 : \Delta t : 5]$, where $\Delta t = 2^{-2}, 2^{-3}, 2^{-4}, \ldots, 2^{-8}$. For each of these Δt values, calculate the error $E = \text{norm}((y_{true} - y_{num}), 2)$ of the numerical method. Plot $\log(\Delta t)$ on the x axis and $\log(E)$ on the y axis. Using polyfit, find the slope of the best fit line through this data. This is the order of Forward Euler's method.

ANSWERS: Save your numerical solution as **column** vectors in A1-A7.dat for $\Delta t = 2^{-2}, 2^{-3}, \ldots, 2^{-8}$ respectively. Save the error values in a **row** vector with seven components in A8.dat. Save the slope of the line in A9.dat.

(b) Solve this ODE numerically using Heun's method:

$$y_{n+1} = y_n + \frac{\Delta t}{2}[f(t_n, y_n) + f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n))]$$

with $t = [0 : \Delta t : 5]$, where $\Delta t = 2^{-2}, 2^{-3}, \ldots, 2^{-8}$. For each of these Δt values, calculate the error $E = \text{norm}((y_{true} - y_{num}), 2)$ of the numerical method. Plot $\log(\Delta t)$ on the x axis and $\log(E)$ on the y axis. Using polyfit, find the slope of the best fit line through this data. This is the order of Heun's method.

ANSWERS: Save your numerical solution as **column** vectors in A10-A16.dat for $\Delta t = 2^{-2}, 2^{-3}, \ldots, 2^{-8}$ respectively. Save the error values in a **row** vector with seven components in A17.dat. Save the slope of the line in A18.dat.

2. Consider the Lorentz differential equations (use ODE45)

$$\frac{dx}{dt} = \sigma(-x + y) + \gamma\cos(t), \qquad \frac{dy}{dt} = rx - y - xz, \qquad \frac{dz}{dt} = -bz + xy.$$

with $b = 8/3$, $\sigma = 10$, $t = [0 : 1 : 100]$, and $x(0) = y(0) = z(0) = 1$.

(a) Solve the system with $\gamma = 0$ and $r = 0.5, 15$, and 28. Compare the dynamics for each of the r values and plot the results using PLOT3 and GRID ON.

ANSWERS: Should be written out as three column matrices A19-A21.dat for $r = 0.5, 15$ and ,28 respectively.

(b) Repeat part (a) with $\gamma = 1$.

ANSWERS: Should be written out as three column matrices A22-A24.dat for $r = 0.5, 15$ and 28 respectively.

3. Consider the boundary value problem:

$$\frac{d^2\phi_n}{dx^2} + [100n(x) - \beta_n]\,\phi_n = 0.$$

The function $n(x)$ is given by

$$n(x) = \begin{cases} 1 - |x|^2 & \text{when } 0 \le |x| \le 1, \\ 0 & \text{when } |x| > 1 \end{cases}$$

with boundary conditions $\phi(\pm L) = 0$. Use $L = 2$ and solve for $x = -L : .1 : L$.

(a) Calculate the first five *normalized* eigenvectors (ϕ_n) and eigenvalues (β_n) using a shooting scheme and ode45 (*Note*: normalization $\int_{-L}^{L} |\phi_n|^2 dx = 1$).

ANSWERS: Eigenvectors should be written out as matrix with five columns in A25.dat. Eigenvalues should be in a **row** vector with five components in A26.dat *Note*: Use the same tolerance level as code on web page.

(b) Calculate the first five *normalized* eigenvectors (ϕ_n) and eigenvalues (β_n) using the direct solve scheme outlined in class. (*Note*: normalization $\int_{-L}^{L} |\phi_n|^2 dx = 1$). You will need the EIG command for this.

ANSWERS: Eigenvectors should be written out as matrix with five columns in A27.dat. Eigenvalues should be in a **row** vector with five components in A28.dat

(*Note*: Each data file is worth 3 points for this problem)