Amath 301 Midterm, Summer 2007
SOLUTIONS

1. (25 pts.)

a. (3 pts) Write the output of the following code. Be sure to note where the semicolons are.

```
for i=1:2:7
  m(i)=3;
end
m
```

`Ans:  m=[3 0 3 0 3 0 3]` Note: the problem was a bit ambiguous. since m(2),m(4) and m(6) were not defined, i also accepted ”error”.

b. (6 pts) Suppose

$$A = \begin{pmatrix} 2 & 7 & 5 & 8 \\ 3 & 9 & 8 & 0 \\ 5 & 0 & 0 & 9 \\ 3 & 4 & 6 & 2 \end{pmatrix}$$

has been entered into MATLAB as the variable `A`. Write the output of the following commands.

i. `A(:,1)`
   ans:
   ```
   A=[2
   3
   5
   3]
   ```

ii. `A(3,2:4)`
   ans: `[0 0 9]`

iii.
   ```
   gg=0
   for j=1:4
     gg=gg+A(j,j);
   end
   gg
   ```
   ans: `gg= 13`

c. (8 pts) Suppose we have a vector of data stored in MATLAB as `x` and the function below saved as stats.m in the current directory:

```
function [numpts,avg]=stats(somevec)

numpts=length(somevec);
avg=sum(somevec)/numpts;
```

Write a command which, using this function, stores the number of components and average of `x` in the variables `n` and `mu`, respectively.

ans: `[n,mu]=stats(x)`

d. (8 pts) Suppose we have a matrix A, and it's LU decomposition has been stored in MATLAB through the command:

```
[L,U,P]=lu(A);
```

Write a MATLAB command which will solve the system Ax=b for x, where x is stored as `x` and b is stored as `b`. Assume that the permutation matrix is NOT equal to the identity matrix.

ans: `x=U\(L\(P*b))`

2. (25 pts.) Consider the following data set:

$$(x, y) = \{(x_i, x_{i+1})\} \quad \text{for } i = 1 \ldots K$$

Note: There is a typo! it should be

$$(x, y) = \{(x_i, y_i)\} \quad \text{for } i = 1 \ldots K$$

a. (2 pts) What is the lowest order polynomial which is guarenteed to go through all the points?

   $(K-1)$th order

b. (7 pts) Write a MATLAB code which will compute the best fit for this polynomial, and evaluate it over the interval $x = 1$ to $x = 3$ in steps of 0.1. You may assume the x values of the data are stored in a vector called `x`, and the y values are in the vector `y`.

```
xp=1:0.1:3;
p=polyfit(x,y,K-1);
yp=polyval(p,xp)
```

c. (3 pts) In general, will this polynomial be a good fit for large K? Why or why not?

   No, it is not a good fit because such a high order polynomial is susceptible to polynomial wiggle, i.e. the function will behave wildy between points.

d. (10 pts) Consider a curve of the form

$$f(x) = Ax\ln x + B.$$

Set up the equations which yield the best fit for this curve to the data in the least-squares sense.

$$E = \sum_{i=1}^{n}(Ax_i \ln x_i + B - y_i)^2$$

$$\frac{\delta E}{\delta A} = \sum_{i=1}^{n} 2(Ax_i \ln x_i + B - y_i)x_i \ln x_i = 0$$

$$\frac{\delta E}{\delta B} = \sum_{i=1}^{n} 2(Ax_i \ln x_i + B - y_i) = 0$$

e. (3 pts) Can the system of equations in part d. be written in the form $Ax = b$? Is so, specify the matrix A and the vector b. If not, why not?

yes:

$$A = \begin{pmatrix} \sum_{i=1}^{n}(x_i \ln x_i)^2 & \sum_{i=1}^{n}(x_i \ln x_i) \\ \sum_{i=1}^{n}(x_i \ln x_i) & \sum_{i=1}^{n} 1 \text{ or } n \end{pmatrix}$$

$$b = \begin{pmatrix} \sum_{i=1}^{n} y_i x_i \ln x_i \\ \sum_{i=1}^{n} y_i \end{pmatrix}$$

note that these equations ARE linear in A and B (that means the terms A and B only appear alone. there are no terms such as A^2 or AB or e^A.)

3. (25 pts.) A spline is a cubic fit between two points. Assume the spline between the points (x_i, y_i) and (x_{i+1}, y_{i+1}) is called $S_i(x)$.

a. (12 pts) What are the four constraints which define splines? You may express your answer a set of equations involving $S_i(x)$ or **clearly** in words.

- The splines go through all the points, $S_i(x_i) = y_i$
- The splines are continuous, $S_i(x_{i+1}) = S_{i+1}(x_{i+1})$
- The splines have continuous derivative, $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$
- The splines have continuous second derivative, $S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$

b. (6 pts) Consider a data set with n points,

$$(x, y) = \{(x_i, y_i)\}, \quad \text{for } i = 1 \ldots n.$$

i. How many splines are needed to fit this data? $n - 1$

ii. How many unknown coefficients does this yield? $4(n-1) = 4n - 4$

iii. How many linear equations are created by the contraints in part a.? $4n - 6$

c. (4 pts) What additional contraints will MATLAB automatically impose when using the spline command with no options? Express the constraints in the form of equations or **clearly** in words.

MATLAB's default condition is the "not-a-knot" condition,

$$S_1'''(x_2) = S_2'''(x_2), S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$$

no points were taken off for mentioning the "clamped spline" condition, as long as you had "not-a-knot" as well.

d. (3 pts) What is the mean-squared error between the spline fit and the data? The error is zero because the spline goes through all the points! one point was given for writing the correct definition of mean sq. error.

4. (25 pts.)

a. (12 pts) Derive the center difference scheme:

$$f'(t) = \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t}$$

using Taylor series expansions. Expand each expression to at least $O(\Delta t^3)$.

$$f(t+\Delta t) = f(t) + \Delta t f'(t) + \frac{\Delta t^2}{2!} f''(t) + \frac{\Delta t^3}{3!} f'''(t) + O(\Delta t)$$

$$f(t-\Delta t) = f(t) - \Delta t f'(t) + \frac{\Delta t^2}{2!} f''(t) - \frac{\Delta t^3}{3!} f'''(t) + O(\Delta t)$$

$$f(t+\Delta t) - f(t-\Delta t) = 2\Delta t f'(t) + \frac{2\Delta t}{3!} f'''(t) + \dots$$

$$\frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} = f'(t) + \frac{\Delta t^2}{3!} f'''(t) + \dots$$

b. (3 pts) What is the truncation error?

$$\frac{\Delta t^2}{3!} f'''(t)$$

c. (10 pts) Suppose we were to use this scheme to differentiate the function:

$$f(x) = \sin 2x \quad \text{on } x = [0, 2\pi].$$

What is the optimal Δt to use? (Express machine epsilon as e_r.) If you were unable to compute the truncation error in part b., use $e_t = \frac{4\Delta t}{7}$.

$$E = e_{round} + e_{trunc}$$

$$|e_{round}| \leq \frac{e_r + e_r}{2\Delta t} \quad |e_{trunc}| \leq \frac{\Delta t^2 M}{3!}$$

where M is the bound on $f'''(t)$:

$$f = \sin 2x,\ f' = 2\cos 2x,\ f'' = -4\sin 2x,\ f''' = -8\cos 2x, \quad \text{so } M = 8.$$

$$\left|\frac{\delta E}{\delta \Delta t}\right| \leq \frac{-e_r}{\Delta t^2} + \frac{2\Delta t * 8}{3!} = 0$$

$$\Delta t = \left(\frac{3e_r}{8}\right)^{1/3}$$

******* Extra Credit ********

a. (2 pts.) What does "MATLAB" stand for?

   Matrix (1pt) Laboratory (1pt)

b. (0 pts.) What doesn't MATLAB stand for?
   whatever you think.....