Solving systems of linear equations

See, in addition to regular course material, sections 2.1-2.5 of our supplemental book Numerical Computing in Matlab by Cleve Moler.

e.g.

$$
\begin{aligned}
10x_1 - 7x_2 &= 7 \\
-3x_1 + 2x_2 + 6x_3 &= 4 \\
5x_1 - x_2 + 5x_3 &= 6
\end{aligned}
$$

- -

$$
\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}
$$

$$
\begin{pmatrix} 10 & -7 & 0 \\ 0 & -0.1 & 6 \\ 0 & 2.5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6.1 \\ \end{pmatrix}
$$

$$
\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ \\ 6.1 \end{pmatrix}
$$

$$
\begin{pmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 6.2 \end{pmatrix}
$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -7 & 0 \\ -3 & 2 & 6 \\ 5 & -1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \\ 7 \end{pmatrix}$$

Suppose can write A as product of lower L and upper U triangular matrices:

$$\mathbf{A} = \mathbf{L}\mathbf{U} \rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \;\rightarrow\; \mathbf{L}\mathbf{U}\mathbf{x} = \mathbf{b}$$

Let

$$\mathbf{y} = \mathbf{U}\mathbf{x} \tag{1}$$

$\rightarrow$

$$\mathbf{L}\mathbf{y} = \mathbf{b} \tag{2}$$

Solve (2) by forward substitution:

$$y_1 = b_1 \tag{3}$$

$$m_{21}y_1 + y_2 = b_2 \tag{4}$$

$$m_{31}y_1 + m_{32}y_2 + y_3 = b_3 \tag{5}$$

Then, solve (1) by backward-substition, exactly as above.

First step in finding L and U: recall matrix-matrix multiplication

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} = \begin{pmatrix} (A_{1,1}B_{1,1} + A_{1,2}B_{2,1}) & (A_{1,1}B_{1,2} + A_{1,2}B_{2,2}) \\ (A_{2,1}B_{1,1} + A_{2,2}B_{2,1}) & (A_{2,1}B_{1,2} + A_{2,2}B_{2,2}) \end{pmatrix}$$

SPECIAL CASE:

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} = \begin{pmatrix} B_{1,1} & B_{1,2} \\ mB_{1,1} + B_{2,1} & mB_{1,2} + B_{2,2} \end{pmatrix}$$

In general, multiplying by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & m & 0 & 1 \end{pmatrix}$$

Example of LU decomposition:

$$
\begin{aligned}
A &= \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{pmatrix} \\
A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 3 \\ 8 & 7 & 9 \end{pmatrix} \\
A &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 8 & 7 & 9 \end{pmatrix} \\
A &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 5 \end{pmatrix} \\
A &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \\
A &= LU
\end{aligned}
$$

MATLAB IMPLEMENTATION: backslash \ operator:

To solve $Ax = b$, using a hierarchy of methods, including LU decomposition and forward / back substitution:

```
x = A \ b
```

```
>> A = [2 1 1 ; 4 3 3 ; 8 7 9]
A =
      2       1       1
      4       3       3
      8       7       9


>> b=[1 ; 2 ; 3]
b =
      1
      2
      3
```

```
\newpage
```

```
>> x=A\b
x =
     0.50000000000000
     0.50000000000000
    -0.50000000000000
```

If have a new b, implement again:

```
>> b=[3 ; 3 ; 3]
b =
      3
      3
      3

>> x=A\b
x =
      3
     -3
      0
```

To generate L, U, P matrices: `[L,U,P]=lu(A)`

```
[L,U,P]=lu(A)

L =
    1.0000         0         0
    0.2500    1.0000         0
    0.5000    0.6667    1.0000


U =
    8.0000    7.0000    9.0000
         0   -0.7500   -1.2500
         0         0   -0.6667

P =
     0     0     1
     1     0     0
     0     1     0
```

```
b =

     3
     3
     3

>> y=L\(P*b)

y =

    3.0000
    2.2500
         0
```

```
>> x=U\y

x =

     3
    -3
     0
```

To generate L, U matrices, where L is a permuted version of a lower-triangular matrix (can still forward-solve in O(N^2))

```
>> [L,U]=lu(A)

L =
    0.2500    1.0000         0
    0.5000    0.6667    1.0000
    1.0000         0         0

U =
    8.0000    7.0000    9.0000
         0   -0.7500   -1.2500
         0         0   -0.6667
```

$LUx = b \rightarrow$ solve $Ly = b$

```
b =
     3
     3
     3
```

```
>> y=L\b
y =
    3.0000
    2.2500
         0
```

solve $Ux = y$

```
>> x=U\y
x =
     3
    -3
     0
```