$\partial F_j/\partial x_j$ are continuous on some interval around t_0.

Before continuing further with this chapter, we need to lay down some preliminaries of *Matrix Theory* and *Linear Algebra*. A matrix is a mathematical object which allows us to manipulate systems in an elegant and efficient fashion: it is the filing cabinet of the math world. A matrix is of the form:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

which denotes an $m \times n$ matrix where the m refers to the number of rows and n to the number of columns. Some important concepts which we introduce are the following:

Transpose: $\mathbf{A}^T = (a_{ij})^T = (a_{ji}) \quad \rightarrow \quad \text{if } \mathbf{A} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix} \text{ then } \mathbf{A}^T = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$

Complex Conjugate: $\overline{\mathbf{A}} = \overline{(a_{ij})} \quad \rightarrow \quad \text{if } \mathbf{A} = \begin{pmatrix} i & 5 \\ 3+i & 6 \end{pmatrix} \text{ then } \overline{\mathbf{A}} = \begin{pmatrix} -i & 5 \\ 3-i & 6 \end{pmatrix}$

Adjoint: $\overline{\mathbf{A}}^T = \mathbf{A}^* \quad \rightarrow \quad \text{if } \mathbf{A} = \begin{pmatrix} i & 5 \\ 3+i & 6 \end{pmatrix} \text{ then } \mathbf{A}^* = \begin{pmatrix} -i & 3-i \\ 5 & 6 \end{pmatrix}.$

These three concepts will be important in utilizing the matrix approach to solving differential equations. Finally, we also introduce the concept of a *square matrix*: $n \times n$, and a *vector*: $n \times 1$ or $1 \times n$. We also have the following important properties of matrices:

1. $\mathbf{A} = \mathbf{B}$ if $a_{ij} = b_{ij}$ for each i and j.

2. Zero matrix $\mathbf{0}$ for which $a_{ij} = 0$ for each i and j.

3. Addition and Subtraction: $\mathbf{A} \pm \mathbf{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} + b_{ij})$.
   - Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
   - Associative: $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$

4. Multiply by a number: $\alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij})$.

5. Matrix multiply: $\mathbf{AB} = \mathbf{C}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

$$\begin{pmatrix} 3 & 2 & 1 \\ 6 & 5 & 0 \\ 1 & 8 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + 2\cdot 0 + 1\cdot 2 \\ 6\cdot 1 + 5\cdot 0 + 0\cdot 2 \\ 1\cdot 1 + 8\cdot 0 + 3\cdot 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

   - Distributive: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
   - Associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
   - NOT Commutative: $\mathbf{AB} \neq \mathbf{BA}$

6. Vectors: $\vec{u}^T\vec{v} = \sum_{i=1}^{n} u_i v_i$

$$\vec{u}^T\vec{v} = (u_1 \; u_2 \; \cdots \; u_n)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = (u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n)$$

- $\vec{u}^T\vec{v} = \vec{v}^T\vec{u}$
- $\vec{u}^T(\vec{v}+\vec{w}) = \vec{u}^T\vec{v} + \vec{u}^T\vec{w}$
- $(\alpha\vec{u})^T\vec{v} = \alpha(\vec{u}^T\vec{v}) = \vec{u}^T(\alpha\vec{v})$

Inner Products: $(\vec{u},\vec{v}) = \sum_{i=1}^{n} u_i\overline{v_i} = \vec{u}^T\overline{\vec{v}}$

- $(\vec{u},\vec{v}) = \overline{(\vec{v},\vec{u})}$
- $(\alpha\vec{u},\vec{v}) = \alpha(\vec{u},\vec{v})$
- $(\vec{u},\alpha\vec{v}) = \overline{\alpha}(\vec{u},\vec{v})$
- $(\vec{u},\vec{v}+\vec{w}) = (\vec{u},\vec{v}) + (\vec{u},\vec{w})$

Vector Magnitudes: $(\vec{u},\vec{u})^{1/2} = \sum_{i=1}^{n} u_i\overline{u_i} = \sum_{i=1}^{n} |u_i|$

Orthogonality: $(\vec{u},\vec{v}) = 0$

7. Identity: $\mathbf{I} = (\delta_{ij})$ ($\delta_{ij} = 1$ for $i = j$ and 0 otherwise) $\rightarrow$ $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

8. Inverse: $\mathbf{AB} = \mathbf{I}$ if $\mathbf{B} = \mathbf{A}^{-1}$ which exists for det($\mathbf{A}$)$\neq$0 (nonsingular).

These properties and identities will be important in manipulating and solving systems of differential equations. They should be thought of as a guide to understanding the systems approach to differential equations.

6.2. Lec. 2. Eigenvalues, Eigenvectors, and Linear Independence

This is the second lecture in which a large number of definitions are discussed. The most important of these ideas relates to what are called *eigenvalues* and *eigenvectors*. But before getting to these, we consider an $n \times n$ system of algebraic equations:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{aligned}$$

which can be solved for the x_i since we have n equations and n unknowns. We can rewrite this in the matrix formalism of the last chapter:

$$\mathbf{A}\vec{x} = \vec{b} \quad \rightarrow \quad \mathbf{A} = (a_{ij}),\; \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\; \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

In such an equation, if $\vec{b} = 0$ the equation is called *homogeneous* and if $\vec{b} \neq 0$ the equation is *non-homogeneous*. This terminology is exactly as in our previous chapters.

There are two interesting cases to consider in solving such an equation as $\mathbf{A}\vec{x} = \vec{b}$. The first case is when

$$\det(\mathbf{A}) \neq 0$$

which from the last section implies that we have an inverse to $\mathbf{A}$ given by $\mathbf{A}^{-1}$. Thus we find upon multiplying our equation $\mathbf{A}\vec{x} = \vec{b}$ through by $\mathbf{A}^{-1}$ on the left that

$$\vec{x} = \mathbf{A}^{-1}\vec{b}$$

since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Thus we can find a solution once we find the inverse. We also note that:

$$\text{if } \vec{b} = 0 \text{ then } \vec{x} = 0$$

giving a trivial solution for the homogeneous case. The second case of interest arises when

$$\det(\mathbf{A}) = 0$$

which implies we have no inverse. This case will be of considerable interest to us in what follows.

A couple of things should be pointed out before proceeding. First, the equation $\mathbf{A}\vec{x} = \vec{b}$ does not have a solution for generic $\vec{b}$. In particular, consider the two equations:

$$\mathbf{A}\vec{x} = \vec{b} \qquad \text{and} \qquad \mathbf{A}^*\vec{y} = 0$$

where we recall that $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$. Taking the inner product of the first equation with respect to $\vec{y}$ gives:

$$(\mathbf{A}\vec{x}, \vec{y}) = (\vec{b}, \vec{y}) .$$

By noting that $(\mathbf{A}\vec{x}, \vec{y}) = (\vec{x}, \mathbf{A}^*\vec{y})$, which is the definition of the adjoint, and that $\mathbf{A}^*\vec{y} = 0$ by definition, we find

$$(\vec{b}, \vec{y}) = 0 .$$

This is known as the *Fredholm Alternative Theorem* and is commonly called a *solvability condition.* It states that $\vec{y}$ must be orthogonal to $\vec{b}$ in order for the equation to make sense. To calculate determinants, we note that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = A_1(B_2C_3 - C_2B_3) - A_2(B_1C_3 - C_1B_3) + A_3(B_1C_2 - C_1B_2) .$$

Since we will mostly be working with 2×2 and 3×3 systems, these are important to know.

Example: Solve

$$x_1 - 2x_2 + 3x_3 = 7$$
$$-x_1 + x_2 - 2x_3 = -5$$
$$2x_1 - x_2 - x_3 = 4$$

To purpose of this example is to illustrate how to manipulate such systems using linear algebra techniques. We begin be rewriting the equations in *augmented matrix form*:

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right)$$

which corresponds to our original equations above. The trick now is to eliminate and solve for the x_i. Acting on each equation of the original system is equivalent to acting on a row of the augmented matrix. So we begin by adding the first and second equations together to generate a new second equation, and we also multiply the first equation by -2 and add it to the third equation to generate a new third equation. This gives

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{array}\right)$$

which can be manipulated further by multiplying the second equation by 3 and adding it to the third

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -4 & -4 \end{array}\right).$$

We can then simplify again by dividing the last equation by -4:

$$\left(\begin{array}{rrr|r} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

This then results in $x_3 = 1$ from the last equation and

$$-x_2 + x_3 = 2 \quad \rightarrow \quad x_2 = -1$$

from the second equation and results in

$$x_1 - 2x_2 + 3x_3 = 7 \quad \rightarrow \quad x_1 = 2$$

from the first equation. In total then, our solution is given by

$$\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

which is in our convenient vector notation.

We now move on to a concept which is already familiar to us, that of linear dependence and independence. If we consider a set of vectors added together:

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = 0$$

Then

- if there exists $c_i \neq 0$ which satisfy this: *linear dependent*
- if the c_i can only be zero: *linear independence*

There is a very simple way to determine whether a set of vectors is linearly independent or not. We can rewrite our equation as:

$$\mathbf{X}\vec{c} = 0 \qquad \text{where} \qquad \mathbf{X} = (\vec{x}_1\ \vec{x}_2\ \cdots\ \vec{x}_n)\ ,\ \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{pmatrix}$$

But this is an equation of the form $\mathbf{A}\vec{x} = \vec{b}$. And we know that in the homogeneous case ($\vec{b} = 0$), that if the determinant of $\mathbf{X}$ is not zero that $\mathbf{X}$ has an inverse and $\vec{c}$ must be zero. Alternatively, if the determinant is zero, then the c_i are not necessarily zero. This gives us:

- if $\det(\mathbf{X}) = 0$: then $\vec{c} \neq 0$ and *linear dependence*
- if $\det(\mathbf{X}) \neq 0$: then $\vec{c} = 0$ and *linear independence*

Thus all we have to do is calculate the determinant to determine linear dependence or independence.

We now reconsider the equation

$$\mathbf{A}\vec{x} = \vec{b}\,.$$

Suppose that the vector $\vec{b}$ is actually $\vec{x}$ times some constant. That is, what if $\vec{b} = \lambda\vec{x}$, then

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

which is called an *eigenvalue problem.* This is very interesting as it implies that we act on the vector $\vec{x}$ with some matrix $\mathbf{A}$ and it simply makes $\vec{x}$ shorter or longer depending on λ. The $\vec{x}$ for which this holds is called an *eigenvector* and the corresponding λ is called the *eigenvalue.* We can rearrange the equation to read

$$(\mathbf{A} - \lambda\mathbf{I})\vec{x} = 0$$

for which we know that $\vec{x} = 0$ if the $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$. However, we are interested in solutions $\vec{x}$ which are not zero. And the only way this can happen is if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0\,.$$

For an $n \times n$ system, this condition yields a polynomial of degree n in λ whose roots are the eigenvalues. Recall that a second-order differential equation can be rewritten as two first order equations. Thus the resulting polynomial is of degree two. In fact, the resulting polynomial is the *characteristic equation* we derived in Chapter 3. The important thing then to determine is whether the eigenvalues are real, complex, or perhaps double roots.

Example: Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

The eigenvalues are determined from

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad \rightarrow \quad \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}$$

which gives the polynomial

$$(1-\lambda)(3-\lambda) + 1 = 0 \quad \rightarrow \quad (\lambda - 2)^2 = 0 \quad \rightarrow \quad \lambda = 2$$

which is a double root. The eigenvectors can be found be recalling that $(\mathbf{A} - \lambda \mathbf{I})\vec{x} = 0$ which gives

$$\begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} \vec{x} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x} = 0$$

which gives $x_1 = -x_2$. So if $x_1 = c$ where c is a constant then

$$\vec{x}_1 = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is the eigenvector.

Finally, we close by introducing a matrix for which the adjoint is exactly the same as the original matrix:

$$\mathbf{A}^* = \mathbf{A}\,.$$

In this case, the matrix $\mathbf{A}$ is said to be *self-adjoint* or *Hermitian.* The properties of the special matrix are that the eigenvalues are all real, there exist n linearly independent eigenvectors which are orthogonal and for a repeated root with multiplicity m, there are m orthogonal eigenvectors which result. Hermitian matrices are very important and arise in a variety of phenomena such as quantum mechanics and electrodynamics.

6.3. Lec. 3. Systems of Differential Equations

To utilize the linear algebra techniques of the first two lectures, we consider the system of differential equations:

$$\vec{x}' = \mathbf{P}(t)\vec{x} + \vec{g}(t)$$

where $\mathbf{P}(t)$ and $\vec{g}(t)$ are continuous on some interval I. As in previous chapters, $\vec{g}(t) \neq 0$ is the non-homogeneous case and $\vec{g}(t) = 0$ is the homogeneous case. The