Exam 1 Answers

Instructions: Clearly answer each of the questions below. Show your work and any formulas you employ. Evaluate any integrals which can be solved by elementary means, and simplify all answers as far as possible. Box your answers.

1. Classify and Solve(if possible), the following differential equations. If there are multiple solution methods available, choose one.(10 pts each)

   (a) $y\ln(x) - y\ln(y) + xy' = 0$

   First order nonlinear, homogeneous

   $$y' = -\frac{y}{x}\ln\left(\frac{y}{x}\right)$$

   $$y(x) = xe^{1+cx}$$

   (b) $xy' = 2y + 3$

   First order linear, Separable

   $$y = cx^2 - 3/2$$

   (c) $xy' - \ln(x)y = \ln(x),\ y(1) = 2$

   First order linear, Separable

   $$y(x) = 3e^{\frac{1}{2}\ln^2(x)} - 1$$

   (d) $\frac{\sin(x)\sin(y)}{\cos^2(x)} + \frac{\cos(y)}{\cos(x)}\frac{dy}{dx} = x$

   First order nonlinear, Exact.

   $$\frac{\sin(y(x))}{\cos(x)} - \frac{1}{2}x^2 = C$$

   (e) $9y'' - 12y' + 4y = 0$

   Second order linear constant coefficient, homogeneous.

   $$y(x) = c_1e^{2x/3} + c_2xe^{2x/3}$$

2. (10 pts) Determine for which values of x and $y(x)$ the differential equation below there exists a solution, and for which values that solution is unique. Sketch and label these regions.

   $$(y')^2 + x^2 + y^2 = 4$$

   Rewriting to the form $y' = f(x, y)$,

   $$y' = \pm\sqrt{4 - x^2 - y^2}$$

   So, solutions only exist when $4 > x^2 + y^2$. This is a disc of radius 2 centered at the origin. Nowhere is the solution unique, with the possible exception of $4 = x^2 + y^2$.

3. (40 pts) Solve the following equation given the initial conditions $x(0) = 1/26$, $\dot{x}(0) = 0$

$$\ddot{x} + 8\dot{x} + 41x = \sin(t) + \cos(t)$$

$$r \in \{-4 \pm 5i\}$$
$$x(t) = c_1 e^{-4t}\sin(5t) + c_2 e^{-4t}\cos(5t) + x_p(t)$$

$$\begin{aligned} \text{UndeterminedCoefficients} : x_p(t) &= a_1 \sin(t) + a_2 \cos(t) \\ x_p'(t) &= a_1 \cos(t) - a_2 \sin(t) \\ x_p''(t) &= -a_1 \sin(t) - a_2 \cos(t) \end{aligned}$$

Convenient Values

$$\begin{aligned} t = 0 &: -a_2 + 8a_1 + 41a_2 = 1 \\ t = \pi/2 &: -a_1 - 8a_2 + 41a_1 = 1 \end{aligned}$$

$$x(t) = c_1 e^{-4t}\sin(5t) + c_2 e^{-4t}\cos(5t) + \frac{3}{104}\sin(t) + \frac{1}{52}\cos(t)$$

$x(0) = 1/26$, $x'(0) = 0$.

$$\begin{aligned} \frac{1}{26} &= c_2 + \frac{1}{52} \\ 0 &= 5c_1 - 4c_2 + \frac{3}{104} \end{aligned}$$

$$x(t) = \frac{1}{104}e^{-4t}\sin(5t) + \frac{1}{52}e^{-4t}\cos(5t) + \frac{3}{104}\sin(t) + \frac{1}{52}\cos(t)$$