Exam 2 — Answers

1. Evaluate the following expressions.

   (a) (8 pts) $\frac{(x * x^3)}{x^5}$

$$\frac{1}{x^5}\int_0^x u^3(x-u)du = \frac{1}{20}$$

   (b) (8 pts) $\int_0^\infty x^3 e^{-6x} dx$

$$\int_0^\infty x^3 e^{-sx} dx = \mathcal{L}x^3 = \frac{3!}{s^4}$$
$$\int_0^\infty x^3 e^{-6x} dx = \frac{3!}{6^4} = \frac{1}{216}$$

   (c) (12 pts) $\mathcal{L}^{-1}\left\{\frac{s+5}{s(s+4)}\right\}$

$$\frac{s+5}{s(s+4)} = F(s) = \frac{5}{4}\frac{1}{s} - \frac{1}{4}\frac{1}{s+4}$$
$$f(x) = \frac{5}{4} - \frac{1}{4}e^{-4x}$$

2. (19 pts) Let $y'' + 4y = f(x)$, $y(0) = -1$, $y'(0) = -2$, where

$$f(x) = \begin{cases} 2, & x \le 2 \\ 4-x, & 2 < x \le 3 \\ 1, & x > 3 \end{cases}.$$

   Find the Laplace Transform of y. Don't invert the transform.

$$f(x) = 2 + (2-x)\,u_2(x) + (x-3)\,u_3(x)$$
$$\mathcal{L}\{f(x)\} = \frac{2}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$
$$s^2Y(s) - s(-1) - (-2) + 4Y(s) = \frac{2}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$
$$(s^2+4)Y(s) = -2 - s + \frac{2}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$
$$\mathcal{L}\{y\} = Y(s) = \frac{e^{-3s} - e^{-2s} + 2s - 2s^2 - s^3}{s^2(s^2+4)}$$

3. (16 pts) Consider $x^2y'' - 4xy' + 6y = x^4$. This is an equidimensional Euler equation, and it can be show that the $C_1x^2 + C_2x^3$ is the general solution to the homogeneous equation. Find a particular solution for the inhomogeneous equation using Lagrange's Variation of Parameters approach.

In standard form, $y'' - 4y'/x + 6y/x^2 = x^2$, so let $y_1(x) = x^2,\ y_2(x) = x^3,\ f(x) = x^2$.

$$y_p(x) = \int_0^x \frac{y_1(u)y_2(x) - y_1(x)y_2(u)}{y_1(u)y_2'(u) - y_1'(u)y_2(u)} f(u)du$$

$$y_1(u)y_2'(u) - y_1'(u)y_2(u) = u^4$$
$$y_p(x) = \int_0^x \frac{u^2x^3 - x^2u^3}{u^4} u^2 du$$
$$y_p(x) = x^3 \int_0^x 1du - x^2 \int_0^x udu$$
$$y_p(x) = \frac{1}{2}x^4$$

4. (13 pts) Classify all the singular points of the equation

$$x(x-3)(x^2 - 7x + 12)y'' - (1 + \frac{1}{x-4})y' + 6y = 0$$

Note first that $x^2 - 7x + 12 = (x-3)(x-4)$, and $1 + 1/(x-4) = (x-3)/(x-4)$. In standard form then,

$$y'' - \frac{(x-3)}{x(x-3)^2(x-4)^2}y' + \frac{6}{x(x-3)^2(x-4)}y = 0$$

The singular points are $x \in \{0, 3, 4\}$. From the definitions, 0 and 3 are regular singular points, and 4 is an irregular singular point.

5. (24 pts) Find a recurrence relation for the Maclaurin series coefficients of

$$y'' - 2xy' + 7xy = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + 7x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 7a_n x^{n+1} = 0$$

$$2a_2x^0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} 7a_n x^{n+1} = 0$$

$$2a_2x^0 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - 2(n-2)a_{n-2}x^{n-2} + 7a_{n-3}x^{n-2} = 0$$

$$a_n = \frac{2(n-2)a_{n-2} - 7a_{n-3}}{n(n-1)}$$