

Amath 351 Exam 3(A tale of love and math)

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Instructions: Clearly answer each of the questions below. Show your work and any formulas you employ. Simplify all answers as far as possible. Box your answers. Orbit equations can be left in an implicit form. Complex eigenvectors should not be simplified.

The setting is Paris, France, shortly before the storming of the Bastille. A young English student named Henry studying at the Sorbonne has had the misfortune of falling in love with the beautiful but reclusive Sophie. By a stroke of luck, Henry is able start a correspondence of letters with the young lady by impersonating a mathematics professor at Oxford. Alas, Henry is a poet, not a mathematician, and has asked for your aid in wooing Sophie.

1. Describe the phase-plane behavior for each of the following linear systems and classify the origin as stable or unstable:

(a) $\frac{d\vec{x}}{dt} = \begin{bmatrix} 5 & 101 \\ -3 & -4 \end{bmatrix} \vec{x}$ $tr = 1, det = -20 + 303 = 283, tr^2/4 < det$ The origin is an unstable(source) spiral point.

(b) $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 11 \\ 11 & 1 \end{bmatrix} \vec{x}$ $tr = 0, det = -122$ The origin is a saddle point(hence, unstable).

(c) $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1/3 & 1 \\ -2 & -213 \end{bmatrix} \vec{x}$ $tr = -213 - 1/3, det = 71 + 2 = 73, tr^2/4 > det$ The origin is a stable node(sink).

2. Find the general solution of $y' - \frac{2}{x}y = 3$. The problem is linear, and can be solved by integrating factors:

$$\mu(x) = e^{\int -2/x dx} = \frac{1}{x^2} \quad (1)$$

$$\frac{d}{dx} (\mu(x)y) = \frac{3}{x^2} \quad (2)$$

$$y(x) = Cx^2 - 3x \quad (3)$$

3. Smitten with her correspondent's prose and grasp of mathematics, Sophie asked that he explain to her Cayley's recent discovery of matrices.

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & -4 & 0 \\ 1 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1-4+1 & 3+8-4 & 2+0+1 \\ 1+2 & 3-4 & 2 \\ -8+2 & 16-8 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 7 & 3 \\ 3 & -1 & 2 \\ -6 & 8 & 2 \end{bmatrix}$$

4. Find the general solution to

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} -9 & 13 \\ -4 & 3 \end{bmatrix} \vec{x}$$

Characteristic Equation: $r^2 + 6r + 25 = 0$

Eigenvalues: $r \in \{-3 \pm 4i\}$

Eigenvectors: $\begin{bmatrix} 3 \mp 2i \\ 2 \end{bmatrix}$

$$\vec{x}(t) = C_1 e^{(-3+4i)t} \begin{bmatrix} 3 - 2i \\ 2 \end{bmatrix} + C_2 e^{(-3-4i)t} \begin{bmatrix} 3 + 2i \\ 2 \end{bmatrix}$$

or

$$\vec{x}(t) = C_1 e^{-3t} \begin{bmatrix} 13 \cos 4t \\ 6 \cos 4t - 4 \sin 4t \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 13 \sin 4t \\ 6 \sin 4t + 4 \cos 4t \end{bmatrix}$$

5. *Corresponding with other English mathematicians, Sophie eventually discovers that Henry is not whom he claims to be. An upset Sophie writes to Henry declaring that if he ever wants her to write to him again, he must meet her on le pont Notre-Dame, and bring along the solution to the following problem. Consider the non-linear system:*

$$\begin{aligned} \dot{x} &= \frac{1}{2}x^2 + y - 1 \\ \dot{y} &= -xy \end{aligned}$$

(a) Carefully sketch and label the x-nullclines and y-nullclines of the system, and find the coordinates of all steady-states.

x-nullcline: $y = 1 - x^2/2$

y-nullclines: $x = 0$ or $y = 0$

Steady-states: $(0, 1)$, $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$

(b) Find the equation for the orbits of this system.

$$\frac{dy}{dx} = \frac{-2xy}{2y + x^2 - 2}$$

The equation is not separable, but is exact.

Orbits: $2xy + y^2 - 2y = C$

(c) Linearize the system near the steady-state with the largest x coordinate, and describe the local behavior of the orbits. (Don't solve the linear system)

$$x = \sqrt{2} + dx$$

$$y = dy$$

$$dx = \sqrt{2}dx + dy$$

$$dy = -\sqrt{2}dy$$

$(\sqrt{2}, 0)$ is a saddle point, since the trace is zero and the determinant is negative. (One positive eigenvalue and one negative eigenvalue)

With your completed answer to Sophie's last problem, along with a letter confessing his love and his ruse to win her heart, Henry rushed through the streets. Alas, in his haste to rendezvous with his love, Henry found himself in the midst of an angry revolutionary mob. Discovering the letters in his possession, one written in English and the other apparently containing secret codes, Henry was denounced as an English spy and within 2 hours, his head lay in a basket, a long distance from his body. Sophie never expected her correspondent to arrive, though, and continued on with her life, eventually marrying an obtuse but dashing officer in Napoleon's army.