

Necessities in Differential Equations

Note: In the following, $f, g,$ and h are functions, while $a, b, c, d, \alpha,$ and β are constants.

1 Algebra

1.1 Basics

- $(a \pm b)^2 = a^2 \pm 2ab + b^2$
- $(a - b)(a + b) = a^2 - b^2$
- $\frac{a/b}{c/d} = \frac{ad}{bc}$
- $ax^2 + bx + c \Rightarrow x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

1.2 Exponents and Exponentials

- $(x^a)^b = x^{ab}$
- $x^a x^b = x^{a+b}$
- $x^{-1} = 1/x$
- $e^{-\infty} = 0, e^0 = 1, e^{\infty} = \infty,$

1.3 Logarithms

- $\log_a(b) = c \Leftrightarrow b = a^c$
- $\ln(ab) = \ln(a) + \ln(b)$
- $\ln(a^b) = b \ln(a)$
- $\ln(e^x) = x, e^{\ln(x)} = x$
- $\ln 0 = -\infty, \ln 1 = 0, \ln \infty = \infty,$

1.4 Trigonometry

- $\cos^2(a) + \sin^2(a) = 1$
- $\tan x = \frac{\sin(x)}{\cos(x)} = \frac{1}{\cot(x)}$
- $\sec x = 1/\cos x, \csc x = 1/\sin x$
- $\sin(-a) = -\sin(a), \cos(-a) = \cos(a)$
- $\sin(x) = \cos(x - \pi/2)$
- $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$
- $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$
- $2\sin(a)\cos(b) = \sin(a - b) + \sin(a + b)$
- $2\sin(a)\sin(b) = \cos(a - b) - \cos(a + b)$
- $2\cos(a)\cos(b) = \cos(a - b) + \cos(a + b)$
- $\sin(0) = \sin(\pi) = 0, \sin(\pi/2) = 1, \sin(3\pi/2) = -1$
- $\cos(0) = 1, \cos(\pi/2) = 0 = \cos(3\pi/2), \cos(\pi) = -1$

1.5 Hyperbolic Functions

- $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
- $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
- $\sinh(0) = 0, \cosh(0) = 1$
- $\cosh^2(x) - \sinh^2(x) = 1$

2 Calculus

Fundamental Theorem:

$$\frac{d}{dx} \int_a^x f(u) du = f(x)$$

Leibniz's Theorem:

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(u, x) du = \int_{g(x)}^{h(x)} \frac{\partial}{\partial x} f(u, x) du + f(h(x), x) \frac{dh}{dx} - f(g(x), x) \frac{dg}{dx}$$

2.1 Differentiation

- $(af + bg)' = af' + bg'$
- Product rule: $(fg)' = fg' + f'g$
- Chain rule: $[f(g(x))]' = f'(g(x))g'(x)$

2.2 Integration

- $\int (af + bg) dx = a \int f dx + b \int g dx$
- Substitution: $\int f(g(x))g'(x) dx = \int f(x) dx$
- Integration by parts:
 $\int f g' dx = fg - \int g f' dx$

2.3 Common Results

- $(x^a)' = ax^{a-1}, a \neq -1$
- $\int x^a dx = x^{a+1}/(a+1) + C$
- $(e^{ax})' = ae^{ax}$
- $\int e^{ax} dx = e^{ax}/a + C$
- $(\sinh x)' = \cosh x$
- $\int \cosh x dx = \sinh x + C$
- $(\cosh x)' = \sinh x$
- $\int \sinh x dx = \cosh x + C$

- $(\sin x)' = \cos x$
- $\int \cos x dx = \sin x + C$
- $(\cos x)' = -\sin x$
- $\int \sin x dx = -\cos x + C$
- $(\tan x)' = 1 + \tan^2 x$
- $\int \sec^2(x) dx = \tan x + C$
- $(\ln x)' = 1/x$
- $\int (1/x) dx = \ln x + C$
- $(\arctan x)' = 1/(1+x^2)$
- $\int 1/(1+x^2) dx = \arctan(x) + C$
- $(\arcsin x)' = 1/\sqrt{1-x^2}$
- $\int 1/\sqrt{1-x^2} dx = \arcsin(x) + C$

2.4 Series

- $f(x+y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} \frac{d^n f(x)}{dx^n}$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\sin x = x - x^3/6 + x^5/120 \dots$
- $\cos x = 1 - x^2/2 + x^4/24 \dots$
- $1/(1-x) = 1 + x + x^2 + x^3 \dots$
- $(x+y)^a = \sum_{n=0}^{\infty} \frac{a!}{n!(a-n)!} x^{a-n} y^n$