INTEGRATION: AN EXECUTIVE REVIEW

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The solution techniques for ordinary differential equations, presented in this course, work by reducing the problem to the simpler problem of evaluating integrals. It is therefore of crucial importance that you know how to evaluate integrals. Our expectation is that you know the integration techniques covered in the calculus series and specifically MATH 125. The standard textbook used in the calculus sequence is Stewart [1]. However, because that is expensive, you may want to refer instead to Mendelson [2] and Ayres [3]. You will find a lot of solved examples, as well as a lot of exercises for practice.

The fact that it is possible to write computer programs that evaluate integrals betrays that there is a systematic theory for evaluating integrals that can be implemented as an algorithm, just as there exist systematic rules for differentiating functions. Unfortunately, most textbooks fail to explain this adequately. This review contains an executive summary of the underlying theory for evaluating all the integrals that we expect you to be able to evaluate.

We distinguish between two methods: the *method of substitution* and *integration by parts*. In both cases there is a diversity of the integral types that can be solved using these methods.

1. The method of substitution

The method of substitution is based on the following theorem.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy \tag{1}$$

It is used to evaluate rational, trigonometric, and a few irrational integrals. A rational integral is one that can be written in the following form

$$I = \int \frac{P(x)}{Q(x)}dx \tag{2}$$

where $P(x)$, $Q(x)$ are polynomials. These integrals are simplified first using *partial fractions*, and then the method of substitution is used to reduce them to one of the following two cases.

$$\int \frac{dx}{x-a} = \ln|x-a| + c, \quad \int \frac{dx}{1+x^2} = \arctan x + c \tag{3}$$

For details see Chapter 33 of [3].

Trigonometric integrals are integrals of the form

$$I = \int R(\sin x, \cos x)dx, \tag{4}$$

where R is an algebraic function. This means that it represents operations that involve only addition, subtraction, multiplication, and division. These integrals can always be evaluated. In some cases you may use trigonometric identities and simplify them sufficiently so that they can be evaluated using the fundamental theorem of calculus. Another strategy that works sometimes is to use trigonometric identities to rewrite the integrals in one of the following forms

$$I = \int F(\sin x)\cos x dx \quad (5)$$
$$I = \int F(\cos x)\sin x dx.$$

Then, you can use the substitutions $y = \sin x$, $y = \cos x$ to eliminate the trigonometric functions.

If nothing else works, there is the following technique by Weierstrass that always works. First, you convert all the trigonometric functions to the same angle and then you convert them all to tangents using the following identities.

$$\sin 2a = \frac{2\tan a}{1+\tan^2 a} \quad (6)$$
$$\cos 2a = \frac{1-\tan^2 a}{1+\tan^2 a}.$$

As a result, the integral is now written as

$$I = \int F(\tan x) dx. \quad (7)$$

This form can always be evaluated using the substitution $y = \tan x$. To see this, note that

$$dy = (1+\tan^2 x)dx = (1+y^2)dx, \quad (8)$$

and therefore

$$dx = \frac{dy}{1+y^2}. \quad (9)$$

In the most general case, when the trigonometric functions are eliminated, you get a rational integral that can be evaluated using partial fractions.

If we allow raising to a rational power in forming functions that we wish to integrate, then we obtain integrals that cannot always be evaluated in terms of elementary functions. These integrals are called *irrational integrals.* There are however techniques for evaluating the following types.

Integrals of the form

$$I = \int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{1/n}\right) dx \quad (10)$$

can be converted into rational integrals using the so-called *rationalizing* substitution

$$y = \left(\frac{ax+b}{cx+d}\right)^{1/n}. \quad (11)$$

To apply the substitution, solve for x first, and then differentiate to obtain an expression for dx in terms of y and dy. A special case which is covered by rationalizing substitution is the following integral

$$I = \int R(x, \sqrt{ax+b})dx, \quad (12)$$

which corresponds to setting $c = 0, d = 1, n = 2$.

Another type of irrational integrals that can always be evaluated are the following

$$
\begin{aligned}
I &= \int R(x, \sqrt{a^2 - x^2})dx \\
I &= \int R(x, \sqrt{x^2 - a^2})dx \qquad (13) \\
I &= \int R(x, \sqrt{x^2 + a^2})dx.
\end{aligned}
$$

In this case we use the trigonometric substitutions $y = a \sin x$, $y = a \sec x$, $y = a \tan x$. This leads to trigonometric integrals that can be evaluated using the Weierstrass technique, if nothing else works.

2. Integration by parts

The method of integration by parts is based on the following theorem.

$$
\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx \qquad (14)
$$

When we apply this result to a specific integral, our intention is to arrange things so that a specific factor $g(x)$ of the integrand is differentiated in the new integral. We say that we're *targeting* $g(x)$. To do that, we have to be able to rewrite the other factor as a derivative of something, that is in the form $f'(x)$. If we can do that, then we can apply integration by parts.

The most common situations are the following. First, we may have integrals of the form

$$
\begin{aligned}
I &= \int P(x) \sin(ax + b)dx \\
I &= \int P(x) \cos(ax + b)dx \qquad (15) \\
I &= \int P(x) \exp(ax + b)dx,
\end{aligned}
$$

where $P(x)$ are polynomials. In this case use integration by parts repeatedly targeting the polynomial factor until it is eliminated. Then, the integral can be evaluated by the fundamental theorem of calculus.

Another case is integrals of the form

$$
\begin{aligned}
I &= \int \exp(ax + b) \sin(cx + d)dx \qquad (16) \\
I &= \int \exp(ax + b) \cos(cx + d)dx.
\end{aligned}
$$

Use integration by parts to target the trigonometric factor twice. That leads to an equation of the form $I = a + bI$. Then you can solve for I. Finally, for integrals of the form

$$
\begin{aligned}
I &= \int f(x) \ln g(x) dx \\
I &= \int f(x) \arctan(g(x)) dx \qquad (17) \\
I &= \int f(x) \arcsin(g(x)) dx
\end{aligned}
$$

use integration by parts to target, not $f(x)$, but the *other* factor. We exploit the fact that differentiating these functions simplifies them.

3. Conclusion

That is all there is to it. You may use a computer to evaluate integrals in order to check your answers when you do your homework. You are expected however to be able to evaluate integrals without the assistance of a computer.

There is a commercial program called *Mathematica* that you may use to evaluate integrals. You can find computers that have that program installed in the MSCC lab. If you have a computer at home running GNU/Linux, you may freely download and install a program called *Macsyma* and use that instead.

References

[1] J. Stewart. *Single Variable Calculus, early transcendentals.* Brooks/Cole Publishing Company, Pacific Grove, 1999.
[2] E. Mendelson. *Beginning Calculus.* Schaum Outline Series, New York, 1997.
[3] F. Ayres and E. Mendelson. *Calculus.* Schaum Outline Series, New York, 1999.