AMATH 351 Homework 1

Due January 14, 2009 (W)

Section1.1 15,16,17,18,19,20

Section1.3 1,2,3,4,5,6,13,14,17,19

Section2.2 6,18,32

*For figures and Problem 30, download them on the course webpage

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In each of the following problem verify that each given function is a solution of the differential equation.

13. $y'' + y' = \sec t, \quad 0 < t < \pi/2; \qquad y = (\cos t) \ln \cos t + t \sin t$

$$\begin{aligned}
y &= \cos t \ln(\cos t) + t \sin t \\
y' &= -\sin t \ln(\cos t) + \cos t \frac{1}{\cos t}(-\sin t) + \sin t + t \cos t \\
&= -\sin t \ln(\cos t) + t \cos t \\
y'' &= -\cos t \ln(\cos t) + \sin t \frac{1}{\cos t} \sin t + \cos t - t \sin t \\
\Longrightarrow y'' + y &= -\cos t \ln(\cos t) + \frac{\sin^2 t}{\cos t} + \cos t - t \sin t + \cos t \ln(\cos t) + t \sin t \\
&= \frac{\sin^2 t + \cos^2 t}{\cos t} \\
&= \frac{1}{\cos t} \\
&= \sec t \qquad (0 < t < \frac{\pi}{2})
\end{aligned}$$

■

14. $y' - 2ty = 1; \qquad y = e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2}$

$$y = e^{t^2} \int_0^t e^{-s^2} ds + e^{t^2}$$

$$
\begin{aligned}
y' &= 2te^{t^2}\int_0^t e^{-s^2}ds + e^{t^2}e^{-t^2} + 2te^{t^2} \\
y' - 2ty &= 2te^{t^2}\int_0^t e^{-s^2}ds + 1 + 2te^{t^2} - 2te^{t^2}\int_0^t e^{-s^2}ds - 2te^{t^2} \\
&= 1
\end{aligned}
$$

■

In the following problem determing the value of r for which the given differential equation has solutions of the form $y = e^{rt}$.

17. $y'' + y' - 6y = 0$

$$
\begin{aligned}
y &= e^{rt} \\
y' &= re^{rt} \\
y'' &= r^2e^{rt}
\end{aligned}
$$

$$
y'' + y' - 6y = r^2e^{rt} + re^{rt} - 6e^{rt} = (r^2 + r - 6)e^{rt} = 0
$$

Since $e^{rt} > 0$, divide e^{rt}on both sides of the equation. We get

$$
\begin{aligned}
r^2 + r - 6 &= 0 \\
i.e.\ (r+3)(r-2) &= 0 \\
\Rightarrow r &= -3 \ or\ 2
\end{aligned}
$$

■

In the following problem determine the value of r for which the given differential equation has solutions of the form $y = t^r \quad for \quad t > 0$.

19. $t^2y'' + 4ty' + 2y = 0$

$$
\begin{aligned}
y &= t^r \qquad t > 0 \\
y' &= rt^{r-1} \\
y'' &= r(r-1)t^{r-2}
\end{aligned}
$$

So

$$
\begin{aligned}
& t^2y'' + 4ty' + 2y \\
=\ & t^2r(r-1)t^{r-2} + 4trt^{r-1} + 2t^r \\
=\ & (r^2 - r)t^r + 4rt^r + 2t^r \\
=\ & (r^2 + 3r + 2)t^r \\
=\ & 0
\end{aligned}
$$

Since $t > 0$, $t^r > 0$, we have $r^2 + 3r + 2 = 0 \Longrightarrow r = -1 \ or \ -2$. ■

Section 2.2

Solve the given differential equation

6. $xy' = (1 - y^2)^{1/2}$

$$xy' = \left(1 - y^2\right)^{\frac{1}{2}}$$

This is a separable equation,

- If $1 - y^2 \neq 0$ and $x \neq 0$, we can divide them,

$$\frac{dy}{\sqrt{1 - y^2}} = \frac{dx}{x}$$

integrate both sides with respect to y and x separately,

$$\begin{aligned} \sin^{-1} y &= \ln|x| + C \\ y &= \sin(\ln|x| + C) \end{aligned}$$

- If $1 - y^2 = 0$, $y = \pm 1$. Plug it back into the equation to check if it's the solution,

$$x \cdot 0 = 0$$

So all the solutions are

$$\begin{aligned} y &= \sin(\ln|x| + C), \qquad x \neq 0 \\ y &= \pm 1 \end{aligned}$$

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In the following problem,

(a) Find the solutionof the given initial value problem in expilicit form.

(b) Plot the graph of the solution.

(c) Determine (at least approximately) the interval in which the solution is defined.

18. $y' = (e^{-x} - e^x)/(3 + 4y), \quad y(0) = 1$

$$y' = \frac{e^{-x} - e^x}{3 + 4y}, \qquad y(0) = 1$$

This is a separable equation,

$$(3 + 4y)\,dy = \left(e^{-x} - e^x\right)dx$$

Integrate both sides,

$$3y + 2y^2 = -e^{-x} - e^x + C$$

Applying initial condition $y(0) = 1$,

$$\begin{aligned} 3 + 2 &= -1 - 1 + C \\ \Rightarrow C &= 7 \end{aligned}$$

Hence we get

$$3y + 2y^2 = -e^{-x} - e^x + 7$$

Rewrite it in the standard form of a quadratic equation,

$$2y^2 + 3y + e^{-x} + e^x - 7 = 0$$

So

$$\begin{aligned} y &= \frac{-3 \pm \sqrt{9 - 4 \times 2 \times (e^{-x} + e^x - 7)}}{4} \\ &= -\frac{3}{4} \pm \frac{1}{4}\sqrt{65 - 8e^{-x} - 8e^x} \end{aligned}$$

Since $y = 1 > 0$, we only choose the "+" sign,

$$y = -\frac{3}{4} + \frac{1}{4}\sqrt{65 - 8e^{-x} - 8e^x}.$$

Function y is defined on the interval which makes

$$\begin{aligned} 65 - 8e^{-x} - 8e^x &> 0 \\ \Leftrightarrow e^x + e^{-x} &< 7 \\ \Leftrightarrow (e^x)^2 - 7e^x + 1 &< 0 \end{aligned}$$

Let $u = e^x$, then it becomes $u^2 - 7u + 1 < 0$, and then

$$\ln\left(\frac{7 - 3\sqrt{5}}{2}\right) < x < \ln\left(\frac{7 + 3\sqrt{5}}{2}\right)$$

Plot graph of the solution

■

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In Problem 32:

(a) Show that the given equation is homogeneous.
(b) Solve the differential equation.
(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

32. $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{2xy}$$

(a) $\frac{dy}{dx} = \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$, the RHS is a function of $\frac{y}{x}$, so this is a homogeneous equation.

(b) Let $v = \frac{y}{x}$, then

$$\begin{aligned} y &= vx \\ \frac{dy}{dx} &= x\frac{dv}{dx} + v \end{aligned}$$

Then the differential equation becomes

$$\begin{aligned} & x\frac{dv}{dx} + v = \frac{1+3v^2}{2v} \\ \Rightarrow \quad & x\frac{dv}{dx} = \frac{1+v^2}{2v} \\ \Rightarrow \quad & \frac{2v}{1+v^2}dv = \frac{1}{x}dx \qquad \textit{when } x \neq 0 \\ \Rightarrow \quad & \ln\left(1+v^2\right) = \ln|x| + C \\ \Rightarrow \quad & 1+v^2 = e^C|x| = \pm e^C x \\ \textit{denote} \quad & C_1 = \pm e^C,\ C_1 \neq 0 \\ \Rightarrow \quad & 1+v^2 = C_1 x \end{aligned}$$

Replace v by $\frac{y}{x}$, we get

$$\begin{aligned} 1 + \frac{y^2}{x^2} &= C_1 x \\ \textit{i.e.}\ x^2 + y^2 &= C_1 x^3 \quad x \neq 0 \end{aligned}$$

(c) Direction field and some integral curves ■