AMATH 351 Homework 2 Keys

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Section2.1 14,16,19,31,40

Section2.4 3,12,15,29

Section2.6 8,21,23,24

Section 2.1

In each of Problems 13 through 20 find the solution of the given initial value problem.

14. $y' + 2y = te^{-2t}, \qquad y(1) = 0$

Multiply $\mu(t) = e^{2t}$, then

$$\begin{aligned} e^{2t}y' + 2e^{2t}y &= t \\ \Rightarrow \left(e^{2t}y\right)' &= t \\ \Rightarrow e^{2t}y &= \frac{t^2}{2} + C \\ \Rightarrow y &= e^{-2t}\left(\frac{t^2}{2} + C\right) \end{aligned}$$

Apply initial condition $y(1) = \left(\frac{1}{2} + C\right)e^{-2} = 0 \Longrightarrow C = -\frac{1}{2}$.

So the solution is

$$y = e^{-2t}\left(\frac{t^2}{2} - \frac{1}{2}\right)$$

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16. $y' + (2/t)y = (\cos t)/t^2, \qquad y(\pi) = 0, \quad t > 0$

Integrating factor $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$. Multiply it to the DE, we get

$$\begin{aligned} y't^2 + 2ty &= \cos t \\ \Rightarrow \left(t^2y\right)' &= \cos t \\ \Rightarrow t^2y &= \sin t + C \\ \Rightarrow y &= t^{-2}\left(\sin t + C\right) \end{aligned}$$

Apply initial condition, we have

$$y(\pi) = \pi^{-2}(\sin \pi + C) = C\pi^2 = 0 \Longrightarrow C = 0$$

Plug C back into the general solution,

$$y(t) = t^{-2} \sin t$$

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19. $t^3 y' + 4t^2 y = e^{-t}, \qquad y(-1) = 0, \quad t < 0$
First write it in the standard form

$$\begin{aligned} y' + \frac{4}{t} y &= t^{-3} e^{-t} \\ \mu(t) &= e^{\int \frac{4}{t} dt} = t^4 \end{aligned}$$

then multiply it,

$$\begin{aligned} t^4 y' + 4t^3 y &= te^{-t} \\ \Rightarrow \left(t^4 y\right)' &= t^{e^{-t}} \\ \Rightarrow t^4 y &= \int te^{-t} dt = -te^{-t} - e^{-t} + C \\ \Rightarrow y &= e^{-4} \left(-te^{-t} - e^{-t} + C\right) \end{aligned}$$

Apply initial condition

$$y(-1) = e^{-4}\left(e - e + C\right) = e^{-4} C = 0 \Longrightarrow C = 0$$

Plug it into the general solution, we get

$$y(t) = -e^{-4-t}(t+1)$$

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31. Consider the initial value problem

$$y' - \frac{3}{2} y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

Integrating factor

$$\mu(t) = e^{\int -\frac{3}{2} dt} = e^{-\frac{3}{2} t}$$

So

$$\begin{aligned}
\left(e^{-\frac{3}{2}t}y\right)' &= \left(3t+2e^{t}\right)e^{-\frac{3}{2}t} = 3te^{-\frac{3}{2}t}+2e^{-\frac{t}{2}} \\
\Rightarrow e^{-\frac{3}{2}t}y &= \int 3te^{-\frac{3}{2}t}dt + 2\int e^{-\frac{t}{2}}dt \\
&= -2\left[te^{-\frac{3}{2}t} - \int e^{-\frac{3}{2}t}dt\right] - 4e^{-\frac{t}{2}} \\
&= -2te^{-\frac{3}{2}t} - \frac{4}{3}e^{-\frac{3}{2}t} - 4e^{-\frac{t}{2}} + C \\
\Rightarrow y(t) &= -2t - \frac{4}{3} - 4e^{t} + Ce^{\frac{3}{2}t}
\end{aligned}$$

Apply initial condition $y(0) = -\frac{4}{3} - 4 - C = y_0 \Longrightarrow C = y_0 + \frac{16}{3}$. So the specific solution is

$$y(t) = -2t - \frac{4}{3} - 4e^{t} + (y_0 + \frac{16}{3})e^{\frac{3}{2}t}$$

where the last term is the dominant term. Thus the critical value of y_0 is the value that makes it 0, i.e.

$$y_0 + \frac{16}{3} = 0 \Longrightarrow y_0 = -\frac{16}{3}$$

- when $y_0 > -\frac{16}{3}$, $y(t) \to -\infty \; as \; t \to +\infty$
- when $y_0 < -\frac{16}{3}$, $y(t) \to +\infty \; as \; t \to +\infty$
- when $y_0 = -\frac{16}{3}$, $y(t) = -2t - \frac{4}{3} - 4e^{t} \to -\infty \; as \; t \to +\infty$ ■

In Problem 40 use the method of Problem 38 to solve the given differential equation.

40. $y' + (1/t)y = 3\cos 2t, \qquad t > 0$

Based on 38, we want to find out $A(t)$ that satisfies

$$\begin{aligned}
A'(t) &= 3\cos 2t \cdot e^{\int \frac{1}{t}dt} = 3t\cos 2t \\
\Rightarrow A(t) &= \int 3t\cos 2t dt = \frac{3}{2}t\sin 2t + \frac{3}{4}\cos 2t + C
\end{aligned}$$

So the general solution is

$$\begin{aligned}
y(t) &= A(t)e^{-\int \frac{1}{t}dt} = \frac{A(t)}{t} \\
&= \frac{3}{2}\sin 2t + \frac{3}{4t}\cos 2t + \frac{C}{t}
\end{aligned}$$

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Section 2.4

In each of Problems 1 through 6 determine (without solving the problem) and interval in which the solution of the given initial value problem is certain to exist.

3. $y' + (\tan t)y = \sin t, \qquad y(\pi) = 0$

This is a first order linear DE. $p(t) = t \tan t$ is continuous on $\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$ where k is an integer. $q(t) = \sin t$ is continuous on $(-\infty, +\infty)$. According to Thm 2.4.1, the solution to this IVP exists on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. ■

In each of Problems 7 through 12 state where in the ty-plane the hypotheses of Teorem 2.4.2 are satisfied.

12. $\frac{dy}{dt} = \frac{(\cot t)y}{1+y}$

Function $f(t, y)$ is continuous in $(k\pi, k\pi + \pi) \times (-\infty, -1)$ and $(k\pi, k\pi + \pi) \times (-1, +\infty)$.

$\frac{\partial f}{\partial y} = \frac{\cot t}{(1+y)^2}$, which is continuous in the same area as that of $f(t, y)$. So Thm 2.4.2 is satisfied when

$$t \neq k\pi,\ k \in \boldsymbol{Z} \textit{ and } y \neq -1$$

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In each of Problems 13 through 16 solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

15. $y' + y^3 = 0, \qquad y(0) = y_0$

$$y' = -y^3$$

This is a separable equation, we need to divide $-y^3$, then let's check the case of $y(x) \equiv 0$. Obviously, it is a solution.

If $y(t) \neq 0$,

$$\begin{aligned} -\frac{2}{y^3}dy &= 2dx \\ \Rightarrow y^{-2} &= 2x + C \\ \Rightarrow y^2 &= \frac{1}{2x + C} \end{aligned}$$

For the initial condition,

- If $y_0 > 0$,
$$y_0^2 = \frac{1}{C} \Rightarrow C = \frac{1}{y_0^2} \Rightarrow y = \frac{y_0}{\sqrt{2xy_0^2 + 1}}$$
- If $y_0 < 0$,
$$y_0^2 = \frac{1}{C} \Rightarrow C = \frac{1}{y_0^2} \Rightarrow y = -\sqrt{\frac{1}{2x + \frac{1}{y_0^2}}} = -\sqrt{\frac{y_0^2}{2xy_0^2 + 1}} = \frac{y_0}{\sqrt{2xy_0^2 + 1}}$$

- If $y_0 = 0$, then y falls into the first case we talked above: $y(x) \equiv 0$.

Now let's analyze the interval on which the solution exists,

- If $y_0 \neq 0$, $2xy_0^2 + 1 > 0 \Rightarrow x > -\frac{1}{2y_0^2} \Rightarrow \left(-\frac{1}{2y_0^2}, +\infty\right)$;
- If $y_0 = 0$, no ristriction on $x \Rightarrow (-\infty, +\infty)$. ■

Bernoulli Equations. Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. The most important such equation has the form

$$y' + p(t)y = q(t)y^n.$$

and is called a Bernoulli equation after Jakob Bernoulli. Problem 27 through 31 deal with equations of this type.

29. $y' = ry - ky^2$, $r > 0$ and $k > 0$.

Substitute $v = y^{-1}$, then $y = \frac{1}{v}$, $\frac{dy}{dt} = -\frac{1}{v^2}\frac{dv}{dt}$. Now the DE becomes

$$\begin{aligned} -\frac{1}{v^2}\frac{dv}{dt} - r\frac{1}{v} &= -k\frac{1}{v^2} \\ \Rightarrow \frac{dv}{dt} + rv &= k \end{aligned}$$

Integrating factor $\mu(t) = e^{rt}$, multiply it to the above equation,

$$\begin{aligned} \left(e^{rt}v\right)' &= ke^{rt} \\ \Rightarrow v(t) &= \frac{k}{r} + Ce^{-rt} \\ \Rightarrow y(t) &= \frac{1}{v(t)} = \frac{r}{k + Cre^{-rt}} \end{aligned}$$

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Section 2.6

Determin whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

8. $(e^x \sin y + 3y)dx - (3x - e^x \sin y)dy = 0$

Check

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{e^x \cos y + 3 - (-3 + e^x \sin y)}{-3x + e^x \sin y} = \frac{e^x \cos y + 6 - e^x \sin y}{-3x + e^x \sin y} \quad \times$$

Check

$$\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} = \frac{-e^x \cos y - 6 + e^x \sin y}{e^x \sin y + 3y} \quad \times$$

So it's not an exact equation. ■

Show that the equations in Problems 19 through 22 are not exact but become exact when multiplied by the given integrating factor. Then solve the equations.

21. $y dx + (2x - ye^y) dy = 0, \qquad \mu(x,y) = y$

Check

$$\frac{\partial P}{\partial y} = 1 \neq \frac{\partial Q}{\partial x} = 2$$

which is not exact. Multiply $\mu(x,y) = y$, the DE becomes

$$y^2 dx + \left(2xy - y^2 e^y\right) dy = 0$$

Check

$$\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x} = 2y$$

which means exact now.So

$$\begin{cases} \frac{\partial \Phi}{\partial x} = y^2 \\ \frac{\partial \Phi}{\partial y} = 2xy - y^2 e^y \end{cases}$$

From the first equation, we have $\Phi(x,y) = xy^2 + \phi(y)$, then

$$\begin{aligned} \frac{\partial \Phi}{\partial y} &= 2xy + \phi'(y) = 2xy - y^2 e^y \\ \Longrightarrow \phi(y) &= -y^2 e^y + 2ye^y - 2e^y \end{aligned}$$

So the final solution is

$$\Phi(x,y) = xy^2 - y^2 e^y + 2ye^y - 2e^y = C$$

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23. Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y) dy.$$

If the integrating factor is $\mu(y)$, we multiply it to the equation and get,

$$\mu M dx + \mu N dy = 0$$

which now should be an exact equation, i.e.

$$\begin{aligned}
\frac{\partial(\mu M)}{\partial y} &= \frac{\partial(\mu N)}{\partial x} \\
M\frac{d\mu}{dy} + \mu\frac{\partial M}{\partial y} &= \mu\frac{\partial N}{\partial x} \\
\frac{1}{\mu}\frac{d\mu}{dy} &= \frac{N_x - M_y}{M} = Q \\
\ln|\mu| &= \int Q(y)dy \\
\mu(y) &= \exp\left(\int Q(y)dy\right)
\end{aligned}$$

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24. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

If the integrating factor is denoted by $\mu(xy)$, multiply it to the DE we should get an exact equation,

$$\mu M dx + \mu N dy = 0$$

then

$$\begin{aligned}
\frac{\partial(\mu M)}{\partial y} &= \frac{\partial(\mu N)}{\partial x} \\
\Leftrightarrow M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial x} &= N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial y} \\
\Leftrightarrow (M_x - N_y)\mu &= N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} \\
&= Ny\mu'(xy) - Mx\mu'(xy) \\
&= (yN - xM)\mu' \\
\Leftrightarrow \frac{\mu'}{\mu} &= \frac{M_x - N_y}{yN - xM} = R \\
\Leftrightarrow \mu &= \exp\left(\int^{xy} R(s)\,ds\right)
\end{aligned}$$

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