AMATH 351 Homework 3 Keys

January 28, 2009

Section3.1 16,21

Section3.2 3,18,22,24

Section3.3 9,21

Section 3.1

16. $4y'' - y = 0, \quad y(-2) = 1,\ y'(-2) = -1$

Characteristic equation

$$4\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm\frac{1}{2}$$

So the general solution is

$$y(t) = C_1 e^{\frac{t}{2}} + C_2 e^{-\frac{t}{2}}$$

Apply given conditions

$$\begin{cases} y(-2) & = C_1 e^{-1} + C_2 e = 1 \\ y'(-2) & = \frac{C_1}{2} e^{-1} - \frac{C_2}{2} e = -1 \end{cases} \Longrightarrow \begin{cases} C_1 & = -\frac{e}{2} \\ C_2 & = \frac{3}{2e} \end{cases}$$

So the specific solution is

$$y(t) = \frac{1}{2} e^{\frac{t}{2}+1} + \frac{3}{2} e^{-\frac{t}{2}-1} \to -\infty \ as \ t \to +\infty$$

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21. $y'' - y' - 2y = 0, \quad y(0) = \alpha,\ y'(0) = 2$

Characteristic equation

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = -1$$

So the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-t}$$

Apply initial conditions:

$$\begin{cases} y(0) & = C_1 + C_2 = \alpha \\ y'(0) & = 2C_1 - C_2 = 2 \end{cases} \Rightarrow \begin{cases} C_1 & = \frac{\alpha+2}{3} \\ C_2 & = \frac{2\alpha-2}{3} \end{cases}$$

So the specific solution is

$$y(t) = \frac{\alpha+2}{3}e^{2t} + \frac{2\alpha-2}{3}e^{-t}$$

If we want $y(t) \to 0 \quad as \quad t \to +\infty$, the coefficient for e^{2t} must be 0, i.e.

$$\frac{\alpha+2}{3} = 0 \Longrightarrow \alpha = -2$$

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Section 3.2

3.

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}\left(e^{-2t} - 2te^{-2t}\right) + 2te^{-4t} = e^{-4t}$$

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18.

$$\begin{aligned} W &= \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = t^2 e^t \\ &= \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = tg' - g \end{aligned}$$

So we get a DE for $g(t)$:

$$tg' - g = t^2 e^t$$

This is a 1st-order linear DE, so we can solve it by integrating factor method (details omitted). The solution is

$$g(t) = t\left(e^t + C\right)$$

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22.

$$y'' + 4y' + 3y = 0, \quad t_0 = 1$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0 \Longrightarrow \lambda_1 = -1,\ \lambda_2 = -3$$

Then the general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{-3t}$$

We want to find out $y_1(t)$ and $y_2(t)$ which satisfy

$$\begin{cases} y_1(1) = 1, & y_1^{'}(1) = 0; \\ y_2(1) = 0, & y_2^{'}(1) = 1. \end{cases}$$

Apply these conditions, we get

$$\begin{cases} C_1 e^{-1} + C_2 e^{-3} & = 1 \\ -C_1 e^{-1} - 3C_2 e^{-3} & = 0 \end{cases} \Longrightarrow \begin{cases} C_1 & = \frac{3e}{2} \\ C_2 & = -\frac{e^3}{2} \end{cases} \Longrightarrow y_3(t) \quad = \quad \frac{3}{2} e^{-t+1} - \frac{1}{2} e^{3-3t}$$

$$\begin{cases} C_1 e^{-1} + C_2 e^{-3} & = 0 \\ -C_1 e^{-1} - 3C_2 e^{-3} & = 1 \end{cases} \Longrightarrow \begin{cases} C_1 & = \frac{e}{2} \\ C_2 & = -\frac{e^3}{2} \end{cases} \Longrightarrow y_3(t) = \frac{1}{2} e^{-t+1} - \frac{1}{2} e^{3-3t}$$

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24.

$$y'' - 2y' + y = 0 \quad y_1(t) = e^t \; y_2(t) = te^t$$

Plug y_1 and y_2 into the equation to check if they are solutions.
For independency,

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t} \neq 0$$

So y_1 and $y_2$form a fundamental set of solutions.

Section 3.3

9.

The Wronskian of two functions is $W(t) = t \sin^2 t$. Are the functions linearly independent or linearly dependent? Why?

Independent. Because W is not always 0.

21.

From Theorem 3.3.2 (Abel's Theorem),

$$W(y_1, y_2)(t) = c \exp\left[-\int -\frac{2}{t^2} dt \right] = ce^{-\frac{2}{t}}$$

So

$$W(y_1, y_2)(2) = ce^{-1} = 3 \Longrightarrow c = 3e$$

Then

$$\begin{aligned} W(y_1, y_2)(t) &= 3e^{1-\frac{2}{t}} \\ W(y_1, y_2)(4) &= 3e^{1-\frac{2}{4}} = 3e^{\frac{1}{2}} \end{aligned}$$

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