AMATH 351 Homework 4

Due Feb 4, 2009

Section3.4 9,20,40

Section3.5 12,20,26

Section3.6 1,14

Section3.7 4,12

Section 3.4
9

$$y'' + 2y' - 8y = 0$$

Characteristic equation is

$$\begin{aligned} \lambda^2 + 2\lambda - 8 &= 0 \\ \lambda &= 2 \quad or \quad -4 \end{aligned}$$

So the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-4t}$$

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20

$$y'' + y = 0, \quad y(\frac{\pi}{3}) = 2, \quad y'(\frac{\pi}{3}) = -4$$

Characteristic equation is

$$\begin{aligned} \lambda^2 + 1 &= 0 \\ \lambda &= \pm i \end{aligned}$$

So general solution is

$$y = C_1 \cos t + C_2 \sin t$$

Apply the given conditions:

$$\begin{cases} \frac{C_1}{2} + \frac{\sqrt{3}}{2} C_2 & = 2 \\ -\frac{\sqrt{3}}{2} C_1 + \frac{C_2}{2} & = -4 \end{cases} \Longrightarrow \begin{cases} C_1 & = 1 + 2\sqrt{3} \\ C_2 & = -2 + \sqrt{3} \end{cases}$$

So

$$y(t) = \left(1 + 2\sqrt{3}\right) \cos t + \left(-2 + \sqrt{3}\right) \sin t$$

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40

$$t^2 y'' + 4ty' + 2y = 0$$

Let $x = \ln t$, then $\frac{dx}{dt} = \frac{1}{t}$,

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} \\ \frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dx}\right) = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2}\frac{dx}{dt} = -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2} \end{aligned}$$

So the differential equation can be written as

$$\begin{aligned} & t^2\left(-\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}\right) + 4t\frac{1}{t}\frac{dy}{dx} + 2y \\ = & \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \end{aligned}$$

which is a constant coeff. 2nd order linear DE. So the characteristic equation is

$$\begin{aligned} \lambda^2 + 3\lambda + 2 &= 0 \\ \Longrightarrow \lambda &= -1 \quad or \quad -2 \end{aligned}$$

So the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}$$

Change back to t, we get

$$\begin{aligned} y(t) &= C_1 e^{-\ln t} + C_2 e^{-2\ln t} \\ &= \frac{C_1}{t} + \frac{C_2}{t^2} \end{aligned}$$

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Section 3.5
12

$$y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$$

The characteristic equation is

$$\lambda^2 - 6\lambda + 9 = 0 \Longrightarrow \lambda = 3$$

So the general solution is

$$y(t) = C_1 e^{3t} + C_2 t e^{3t}$$

Apply initial conditions, we have

$$\begin{aligned} y(0) &= C_1 = 0 \\ y'(0) &= 3C_1 + C_2 = 2 \end{aligned}$$

So

$$C_1 = 0, \quad C_2 = 2$$

then the solution to the initial value problem is

$$y(t) = 2te^{3t}.$$

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20
(a) The characteristic equation is

$$\begin{aligned} r^2 + 2ar + a^2 &= 0 \\ \Longrightarrow (r+a)^2 &= 0 \\ r_1 = r_2 &= -a \end{aligned}$$

So one solution of the equation is e^{-at}.
(b) According to Abel's Theorem,

$$W(t) = c_1 e^{-\int 2a dt} = c_1 e^{-2at}$$

where c_1 is an arbitary constant.
(c) Let $y_1(t) = e^{-at}$, from part (b), we have

$$y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-at}y_2'(t) + ae^{-at}y_2(t) = c_1 e^{-2at}$$

So the DE for $y_2(t)$ is

$$y_2'(t) + ay_2(t) = c_1 e^{-at}$$

This is a first-order linear DE, the integrating factor is

$$\begin{aligned}
\mu(t) &= e^{\int a\,dt} = e^{at} \\
\Longrightarrow \left(e^{at}y_2\right)' &= c_1 \\
\Longrightarrow e^{at}y_2 &= c_1 t \\
\Longrightarrow y_2 &= c_1 t e^{-at}
\end{aligned}$$

Since we only need to find one independent solution, let $c_1 = 1$ we get $y_2(t) = te^{-at}$. ■

26 Use reduction of order to solve

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t$$

Let $y = v(t)y_1(t)$, and according to page 171 of the book, we have

$$\begin{aligned}
y_1v'' + (2y_1' + py_1)\,v' &= 0 \\
tv'' + \left(2 - \frac{t+2}{t} \times t\right) v' &= 0 \\
tv'' - tv' &= 0 \\
v'' - v' &= 0
\end{aligned}$$

This is a first order differential equation of v',

$$\begin{aligned}
\frac{d(v')}{v'} &= dt \\
\ln v' &= t \\
v' &= e^t \\
v &= e^t
\end{aligned}$$

So the second solution is $y_2(t) = vy_1 = te^t$. ■

Section 3.6

1

$$y'' - 2y' - 3y = 3e^{2t}$$

For homogeneous equation $y'' - 2y' - 3y = 0$, the characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$, so

$$\lambda_1 = 3, \quad \lambda_2 = -1$$

The homogeneous solution is

$$y_H = C_1e^{3t} + C_2e^{-t}$$

Then for nonhomogeneous solution, assume the form to be $y_P = Ae^{2t}$, then plug it into the DE,

$$\begin{aligned} 4Ae^{2t} - 2\times 2Ae^{2t} - 3Ae^{2t} &= 3e^{2t} \\ -3A &= 3 \\ A &= -1 \end{aligned}$$

So the particular solution is $y_P = -e^{2t}$. Then the general solution of the given differential equation is

$$y = C_1 e^{3t} + C_2 e^{-t} - e^{2t}$$

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14

$$y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$$

Firstly find the homogeneous solution

$$\begin{aligned} \lambda^2 + 4 &= 0 \\ \lambda &= \pm 2i \end{aligned}$$

So the homogeneous solution is

$$y_H = C_1 \cos 2t + C_2 \sin 2t$$

For the nonhomogeneous equation, split it into two equations and solve separately

$$\begin{cases} y'' + 4y &= t^2 \\ y'' + 4y &= 3e^t \end{cases}$$

For the first equation, assume $y_{P1} = At^2 + Bt + C$ and plug it back to the given DE,

$$\begin{aligned} 2A + 4At^2 + 4Bt + 4C &= t^2 \\ \Longrightarrow \begin{cases} A &= \frac{1}{4} \\ B &= 0 \\ C &= -\frac{1}{8} \end{cases} \end{aligned}$$

So $y_{P1} = \frac{1}{4}t^2 - \frac{1}{16}$.

For the second equation, assume $y_{P2} = Ae^t$ and plug it back into the given DE,

$$\begin{aligned} Ae^t + 4Ae^t &= 3e^t \\ A &= \frac{3}{5} \end{aligned}$$

So $y_{P2} = \frac{3}{5}e^t$.

Hence the general solution is

$$y(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

Apply initial conditions

$$\begin{cases} C_1 - \frac{1}{8} + \frac{3}{5} & = 0 \\ 2C_2 + \frac{3}{5} & = 2 \end{cases} \Longrightarrow \begin{cases} C_1 & = -\frac{19}{40} \\ C_2 & = \frac{7}{10} \end{cases}$$

So the solution to the IVP is $y(t) = -\frac{19}{40}\cos 2t + \frac{7}{10}\sin 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$. ■

Section 3.7

4

$$4y'' - 4y' + y = 16e^{t/2}$$

Standard form is

$$y'' - y' + \frac{1}{4}y = 4e^{\frac{t}{2}}$$

By solving the homogeneous equation, we get two independent solutions

$$y_1 = e^{\frac{t}{2}}, \quad y_2 = te^{\frac{t}{2}}$$

So the Wronskian is given by

$$W(t) = \begin{vmatrix} e^{\frac{t}{2}} & te^{\frac{t}{2}} \\ \frac{1}{2}e^{\frac{t}{2}} & e^{\frac{t}{2}} + \frac{t}{2}e^{\frac{t}{2}} \end{vmatrix} = e^t + \frac{t}{2}e^t - \frac{t}{2}e^t = e^t$$

Then according to the formulas, we have

$$\begin{aligned} u_1 &= -\int \frac{4e^{\frac{t}{2}}te^{\frac{t}{2}}}{e^t}dt = -2t^2 \\ u_2 &= \int \frac{4e^{\frac{t}{2}}e^{\frac{t}{2}}}{e^t}dt = 4t \end{aligned}$$

So the particular solution is $y_P(t) = -2t^2e^{\frac{t}{2}} + 4t^2e^{\frac{t}{2}} = 2t^2e^{\frac{t}{2}}$.

12

$$y'' + 4y = g(t)$$

The two independent homogeneous solutions are

$$y_1(t) = \cos 2t, \quad y_2(t) = \sin 2t$$

Then the Wronskian is

$$W(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2$$

Then

$$\begin{aligned} u_1(t) &= -\int \frac{g(t)\sin 2t}{2} dt \\ u_2(t) &= \int \frac{g(t)\cos 2t}{2} dt \end{aligned}$$

So the general solution is

$$\begin{aligned} y(t) &= C_1 \cos 2t + C_2 \sin 2t - \cos 2t \int \frac{g(s)\sin 2s}{2} ds + \sin 2t \int \frac{g(s)\cos 2s}{2} ds \\ &= C_1 \cos 2t + C_2 \sin 2t + \frac{1}{2} \int g(s)\left(\sin 2t \cos 2s - \cos 2t \sin 2s\right) ds \\ &= C_1 \cos 2t + C_2 \sin 2t + \frac{1}{2} \int g(s) \sin\left(2t - 2s\right) ds \end{aligned}$$

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