AMATH 351 Homework 5

Due Feb 18, 2009

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Section 7.1
6

$$u'' + p(t)u' + q(t)u = g(t), \quad u(0) = u_0, \quad u'(0) = u'_0$$

Let $u_1 = u, \quad u_2 = u'$, then the differential eqn can be written as

$$u'_2 + p(t)u_2 + q(t)u_1 \quad = \quad g(t)$$

So the initial value problem is equivalent to

$$\begin{cases} u'_1 & = u_2 \\ u'_2 & = -q(t)u_1 - p(t)u_2 + g(t) \end{cases} \quad u_1(0) = u_0, \quad u_2(0) = u'_0$$

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Section 7.2
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$$
\begin{aligned}
&\left(\begin{array}{cccccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \\
\rightarrow &\left(\begin{array}{cccccc} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \\
\rightarrow &\left(\begin{array}{cccccc} 1 & 0 & -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \\
\rightarrow &\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \\
\rightarrow &\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right)
\end{aligned}
$$

So the inverse matrix is

$$
\left(\begin{array}{ccc} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} \end{array}\right)
$$

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$$
A(t) = \left(\begin{array}{ccc} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{array}\right), \quad B(t) = \left(\begin{array}{ccc} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{array}\right)
$$

(a)

$$
\begin{aligned}
A + 3B &= \left(\begin{array}{ccc} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{array}\right) + 3 \times \left(\begin{array}{ccc} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{array}\right) \\
&= \left(\begin{array}{ccc} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{array}\right) + \left(\begin{array}{ccc} 6e^t & 3e^{-t} & 9e^{2t} \\ -3e^t & 6e^{-t} & 3e^{2t} \\ 9e^t & -3e^{-t} & -3e^{2t} \end{array}\right) \\
&= \left(\begin{array}{ccc} 7e^t & 5e^{-t} & 8e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & -e^{2t} \end{array}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
AB &= \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} \cdot \begin{pmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{pmatrix} \\
&= \begin{pmatrix} 2e^{2t} - 2 + 3e^{3t} & 1 + 4e^{-2t} - e^t & 3e^{3t} + 2e^t - e^{4t} \\ 4e^{2t} - 1 - 3e^{3t} & 2 + 2e^{-2t} + e^t & 6e^{3t} + e^t + e^{4t} \\ -2e^{2t} - 3 + 6e^{3t} & -1 + 6e^{-2t} - 2e^t & -3e^{3t} + 3e^t - 2e^{4t} \end{pmatrix}
\end{aligned}
$$

(c)

$$
\begin{aligned}
dA/dt &= \frac{d}{dt} \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} \\
&= \begin{pmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{pmatrix}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\int_0^1 A(t)dt &= \int_0^1 \begin{pmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{pmatrix} dt \\
&= \begin{pmatrix} e^1 - 1 & -2e^{-1} + 2 & \frac{1}{2}e^2 - \frac{1}{2} \\ 2e^1 - 2 & -e^{-1} + 1 & -\frac{1}{2}e^2 + \frac{1}{2} \\ -e^1 + 1 & -3e^{-1} + 3 & e^2 - 1 \end{pmatrix}
\end{aligned}
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22
On the left side,

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\mathbf{x}' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \times 2e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}
$$

on the fight side,

$$
\begin{aligned}
\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} &= \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} \\
&= \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}
\end{aligned}
$$

so left = right, verifying that $\mathbf{x}$ satisfies the differential equation.
Section 7.3
22

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{pmatrix} = -(\lambda-1)(\lambda-2)(\lambda-3) = 0,$$

so eigenvalues are $\lambda_1 = 1, \quad \lambda_2 = 2, \lambda_3 = 3.$

For $\lambda_1 = 1$, we have

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 2$, we have

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = 3$, we have

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

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Section 7.5

2

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \mathbf{x}$$

Denote $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$, then $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{pmatrix} =$ $(\lambda + 1)(\lambda + 2) = 0$, then $\lambda_1 = -1, \quad \lambda_2 = -2.$

For $\lambda_1 = -1$,

$$\begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = 0 \Longrightarrow \xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = -2$,

$$\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = 0 \Longrightarrow \xi_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

So the solution is

$$\mathbf{x(t)} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t}$$

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$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}$$

Denote the matrix as A.

$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{pmatrix} = -(\lambda+1)(\lambda-1)(\lambda-4) = 0$, then $\lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 4.$

For $\lambda_1 = -1$,

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \xi_1 = 0 \Longrightarrow \xi_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 1$,

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \xi_2 = 0 \Longrightarrow \xi_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 4$,

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \xi_3 = 0 \Longrightarrow \xi_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So the general solution of the given system of equations is

$$\mathbf{x(t)} = C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + C_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}$$

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$$A = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix}$$

$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & -1 \\ -\alpha & -1-\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 1 - \alpha = 0$. The roots to this equation are

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4(1-\alpha)}}{2} = -1 \pm \sqrt{\alpha}$$

(a) For $\alpha = 0.5$. $\lambda_1 = -1 + \frac{\sqrt{2}}{2}, \quad \lambda_2 = -1 - \frac{\sqrt{2}}{2}$. The equilibrium point at the origin is a node.

$$\mathbf{x} = C_1 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{\left(-1+\frac{\sqrt{2}}{2}\right)t} + C_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{\left(-1-\frac{\sqrt{2}}{2}\right)t}$$

(Calculation for eigenvectors omitted.)

(b) For $\alpha = 2$. $\lambda_1 = -1 + \sqrt{2}$, $\lambda_2 = -1 - \sqrt{2}$. The equilibrium point at the origin is a saddle point.

$$\mathbf{x} = C_2 \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} e^{(-1+\sqrt{2})t} + C_2 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{(-1-\sqrt{2})t}$$

(Calculation for eigenvectors omitted.)

(c) The transitional state happens when the positive root becomes negative, so

$$-1 + \sqrt{\alpha} = 0 \Longrightarrow \alpha = 1$$

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