AMATH 351 Homework 7

Due March 6, 2009

Section5.1 3, 9, 13

Section5.2 4, 12

Section5.4 6

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Section5.6 6

Section5.7 7

Section 5.1

3.

Determine the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

For the series to be convergent,

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)}}{(n+1)!}}{\frac{x^{2n}}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| = 0$$

As indicated above, the ratio goes to 0 for all the finite values of x, so the radius of convergence is ∞.■

9.

$$\begin{aligned}\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\end{aligned}$$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\times\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty}\left|-\frac{x^2}{(2n+3)(2n+2)}\right| = 0$$

for all the finite values of x, the radius of convergence is ∞.■

13.

$$\begin{aligned}\ln x &= \ln(1+x-1)\\ &= (x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots\\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}\end{aligned}$$

By ratio test

$$\lim_{n\to\infty}\left|(-1)^{n+2}\frac{(x-1)^{n+1}}{n+1}\times(-1)^{n+1}\frac{n}{(x-1)^n}\right| = \lim_{n\to\infty}\left|-\frac{n}{n+1}(x-1)\right| = |x-1|<1$$

So radius of convergence is 1. ■

Section 5.2

4.

$$y''+k^2x^2y=0,\quad x_0=0,\quad k\ a\ constant$$

Since $x_0=0$ is a regular point of the differential equation, we assume

$$\begin{aligned}y &= \sum_{n=0}^{\infty}a_n x^n\\ y' &= \sum_{n=1}^{\infty}n a_n x^{n-1}\\ y'' &= \sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}\end{aligned}$$

Substitue into the DE, we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} k^2 a_n x^{n+2} = 0$$

Shift the index for the both of the series, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} k^2 a_{n-2} x^n = 0$$

The recurrence relation is $(n+2)(n+1) a_{n+2} + k^2 a_{n-2} = 0 \quad n = 2, 3, 4, \ldots$

- for constant term, $2 \cdot 1 \cdot a_2 = 0$;
- for x, $3 \cdot 2 \cdot a_3 = 0$;
- for x^2, $4 \cdot 3 \cdot a_4 + k^2 a_0 = 0 \Rightarrow a_4 = -\frac{k^2}{4 \cdot 3} a_0$;
- for x^3, $5 \cdot 4 \cdot a_5 + k^2 a_1 = 0 \Rightarrow a_5 = -\frac{k^2}{5 \cdot 4} a_1$;
- $\cdots$

So we get the general formula:

$a_{4n+2} = a_{4n+3} = 0, \quad n = 0, 1, 2, \ldots$

$a_{4n} = (-1)^n \frac{k^{2n}}{(4n)(4n-1)(4n-4)(4n-5)\cdots 8 \cdot 7 \cdot 4 \cdot 3} a_0, \quad n = 1, 2, \ldots$

$a_{4n+1} = (-1)^n \frac{k^{2n}}{(4n+1)(4n)(4n-3)(4n-4)\cdots 9 \cdot 8 \cdot 5 \cdot 4} a_1, \quad n = 1, 2, \ldots$

Plug them into the fomal form of solution, we get

$$\begin{aligned} y &= a_0 \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{k^{2n}}{(4n)(4n-1)(4n-4)(4n-5)\cdots 8 \cdot 7 \cdot 4 \cdot 3} x^{4n} \right] \\ &+ a_1 \left[x + \sum_{n=1}^{\infty} (-1)^n \frac{k^{2n}}{(4n+1)(4n)(4n-3)(4n-4)\cdots 9 \cdot 8 \cdot 5 \cdot 4} x^{4n+1} \right] \end{aligned}$$

■

12.

$$(1-x) y" + x y' - y = 0, \quad x_0 = 0$$

Since $x_0 = 0$ is a regular point to the DE, we guess the form of solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}
\end{aligned}
$$

Substitute them in the DE,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Shift the index for the first two series, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

So the recurrence relation is

$$
\begin{aligned}
2 \cdot 1 \cdot a_2 - a_0 &= 0; \\
(n+2)(n+1) a_{n+2} - (n+1) n a_{n+1} + (n-1) a_n &= 0. \quad n = 1, 2, 3, \ldots
\end{aligned}
$$

Then the first few terms are

- $a_2 = \frac{1}{2 \cdot 1} a_0;$
- $3 \cdot 2 \cdot a_3 - 2 \cdot 1 \cdot a_2 + 0 \cdot a_1 = 0 \Rightarrow a_3 = \frac{1}{3} a_2 = \frac{1}{6} a_0;$
- $4 \cdot 3 \cdot a_4 - 3 \cdot 2 \cdot a_3 + 1 \cdot a_2 = 0 \Rightarrow a_4 = \frac{a_0}{24};$
- $5 \cdot 4 \cdot a_5 - 4 \cdot 3 \cdot a_4 + 2 \cdot a_3 = 0 \Rightarrow a_5 = \frac{3}{5} a_4 - \frac{1}{10} a_3 = \frac{a_0}{120};$
- $\cdots$

So the general solution is

$$y = a_0 \left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots + \frac{x^n}{n!} + \cdots \right] + a_1 x$$

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Section 5.4

6. Bessel equation

$$x^2y" + xy' + \left(x^2 - \nu^2\right) y = 0$$

Obviously, $x = 0$ is a singular point of the DE. Now let's check if it's regular singular point or not,

$$\begin{aligned} \lim_{x\to 0} \frac{x}{x^2} \times x &= 1 \\ \lim_{x\to 0} \frac{x^2 - \nu^2}{x^2} \times x^2 &= -\nu^2 \end{aligned}$$

Both of the limits are finite, so $x = 0$ is a regular singular point.

Section 5.5

10.

$$(x-2)^2 y" + 5(x-2) y' + 8y = 0$$

Make a substitutuion $t = x - 2$, then the DE becomes

$$t^2y" + 5ty' + 8y = 0$$

This is an Euler equation

$$\begin{aligned} r(r-1) + 5r + 8 &= 0 \\ \Rightarrow r^2 + 4r + 8 &= 0 \\ \Rightarrow r_{1,2} &= -2 \pm 2i \end{aligned}$$

So the general solution valid in any interval not including $x = 0$ is

$$y = |x|^{-2}\left[c_1 \cos(2\ln|x|) + c_2 \sin(2\ln|x|)\right]$$

■

Section 5.6

6.

$$x^2y" + xy' + (x-2) y = 0$$

First check that $x = 0$ is a regular singular point.(proof omitted)
We assume the solution's form to be

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+\rho} \\ y' &= \sum_{n=0}^{\infty} a_n (n+\rho) x^{n+\rho-1} \\ y" &= \sum_{n=0}^{\infty} a_n (n+\rho)(n+\rho-1) x^{n+\rho-2} \end{aligned}$$

Plug them into the DE, we get

$$\sum_{n=0}^{\infty} a_n (n+\rho)(n+\rho-1) x^{n+\rho} + \sum_{n=0}^{\infty} a_n (n+\rho) x^{n+\rho} + \sum_{n=0}^{\infty} a_n x^{n+\rho+1} - \sum_{n=0}^{\infty} 2a_n x^{n+\rho} = 0$$

Change the index for the third series, we get

$$\sum_{n=0}^{\infty} a_n x^{n+\rho+1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+\rho}$$

For $n = 0$, we have

$$\begin{aligned} a_0 \rho (\rho - 1) + a_0 \rho - 2a_0 &= 0 \\ i.e. \quad \rho^2 - 2 &= 0 \\ \Rightarrow \rho &= \pm\sqrt{2} \end{aligned}$$

From the DE we get the recurrence relation

$$\begin{aligned} (n+\rho)(n+\rho-1) a_n + (n+\rho) a_n + a_{n-1} - 2a_n &= 0 \\ i.e. \quad \left(n^2 + 2\rho n + \rho^2 - 2\right) a_n + a_{n-1} &= 0 \end{aligned}$$

For $\rho = \sqrt{2}$, the recurrence relation is

$$n\left(n + 2\sqrt{2}\right) a_n + a_{n-1} = 0, \quad n = 1, 2, 3, \ldots$$

So

$$\begin{aligned} a_1 &= -\frac{a_0}{1 \cdot \left(1 + 2\sqrt{2}\right)}, \\ a_2 &= \frac{a_0}{2 \cdot \left(2 + 2\sqrt{2}\right) \cdot 1 \cdot \left(1 + 2\sqrt{2}\right)} \\ &\cdots \\ a_n &= (-1)^n \frac{a_0}{n! \left(n + 2\sqrt{2}\right)\left(n - 1 + 2\sqrt{2}\right) \cdots \left(1 + 2\sqrt{2}\right)} \end{aligned}$$

Then one solution is

$$\begin{aligned} y &= x^{\sqrt{2}} - \frac{x^{1+\sqrt{2}}}{1 \cdot \left(1+2\sqrt{2}\right)} + \frac{x^{2+\sqrt{2}}}{2!\left(2+2\sqrt{2}\right)\left(1+2\sqrt{2}\right)} + \cdots \\ &+ (-1)^n \frac{x^{n+\sqrt{2}}}{n!\left(n+2\sqrt{2}\right)\left(n-1+2\sqrt{2}\right)\cdots\left(1+2\sqrt{2}\right)} + \cdots \end{aligned}$$

For $\rho = -\sqrt{2}$, the recurrence relation is

$$n\left(n-2\sqrt{2}\right)a_n + a_{n-1} = 0, \quad n = 1, 2, 3, \ldots$$

Then

$$\begin{aligned} a_1 &= -\frac{a_0}{1 \cdot \left(1-2\sqrt{2}\right)}, \\ a_2 &= \frac{a_0}{2 \cdot \left(2-2\sqrt{2}\right) \cdot 1 \cdot \left(1-2\sqrt{2}\right)} \\ &\cdots \\ a_n &= (-1)^n \frac{a_0}{n!\left(n-2\sqrt{2}\right)\left(n-1-2\sqrt{2}\right)\cdots\left(1-2\sqrt{2}\right)} \end{aligned}$$

Then the other solution corresponding to $\rho = -\sqrt{2}$ is

$$\begin{aligned} y &= x^{-\sqrt{2}} - \frac{x^{1-\sqrt{2}}}{1 \cdot \left(1-2\sqrt{2}\right)} + \frac{x^{2-\sqrt{2}}}{2!\left(2-2\sqrt{2}\right)\left(1-2\sqrt{2}\right)} + \cdots \\ &+ (-1)^n \frac{x^{n-\sqrt{2}}}{n!\left(n-2\sqrt{2}\right)\left(n-1-2\sqrt{2}\right)\cdots\left(1-2\sqrt{2}\right)} + \cdots \end{aligned}$$

■

Section 5.7

7.

$$x^2 y\text{"} + \frac{1}{2}\left(x + \sin x\right) y' + y = 0$$

$x = 0$ is a singular point, let's check if it's RSP.

$$\begin{aligned} \lim_{x\to 0} \frac{1}{2} \times \frac{x + \sin x}{x^2} \times x &= \lim_{x\to 0} \frac{1}{2} + \frac{\sin x}{2x} = 1 \\ \lim_{x\to 0} \frac{1}{x^2} \times x^2 &= 1 \end{aligned}$$

So it is a RSP.

So the indicial equation is

$$r\left(r-1\right)+r+1=0\Rightarrow r^{2}+1=0\Rightarrow r=\pm i$$

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