AMATH 351 Homework 8

No due day, keep it for review.

Section6.1 13, 19

Section6.2 4, 7, 11, 16, 23

Section 6.1
13. This was the example showed in class.
19.

$$\mathcal{L}\left\{t^2 \sin at\right\}$$

Since $\mathcal{L}\left\{\sin at\right\} = \frac{a}{s^2+a^2}, \quad s > 0$, according to the last row of the table on page 319,

$$\begin{aligned}
\mathcal{L}\left\{t^2 \sin at\right\} &= \mathcal{L}\left\{(-t)^2 \sin at\right\} \\
&= \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) \\
&= \frac{d}{ds}\left(-\frac{2as}{\left(s^2+a^2\right)^2}\right) \\
&= -\frac{2a}{\left(s^2+a^2\right)^2} + \frac{8as^2}{\left(s^2+a^2\right)^3} \\
&= \frac{2a\left(3s^2-a^2\right)}{\left(s^2+a^2\right)^3}, \quad s > 0
\end{aligned}$$

■

Section 6.2
4.

$$\begin{aligned}
\frac{3s}{s^2-s-6} &= \frac{3s}{(s-3)(s+2)} \\
&= \frac{A}{s-3} + \frac{B}{s+2}
\end{aligned}$$

Multiply $(s-3)(s+2)$ on both sides of the equation, we get

$$A(s+2)+B(s-3) \quad = \quad 3s$$

then match the coefficients,

$$\begin{cases} A+B &= 3 \\ 2A-3B &= 0 \end{cases} \Longrightarrow \begin{cases} A &= \frac{9}{5} \\ B &= \frac{6}{5} \end{cases}$$

So

$$\mathcal{L}^{-1}\left\{\frac{3s}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{9}{5}\frac{1}{s-3}+\frac{6}{5}\frac{1}{s+2}\right\} = \frac{9}{5}e^{3t}+\frac{6}{5}e^{-2t}.$$

■

7.

$$\frac{2s+1}{s^2-2s+2} = \frac{2(s-1)+3}{(s-1)^2+1} = 2\frac{s-1}{(s-1)^2+1}+3\frac{1}{(s-1)^2+1}$$

So

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-2s+2}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}+3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} \\ &= 2e^t\cos t+3e^t\sin t\end{aligned}$$

■

11.

$$y" - y' - 6y = 0, \quad y(0) = 1,\ y'(0) = -1$$

Take the Lapace Transform of the equation

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 6Y(s) = 0$$

Make use of the conditions and simplify, we get

$$\begin{aligned}\left(s^2-s-6\right)Y(s) &= s-2 \\ Y(s) &= \frac{s-2}{s^2-s-6}\end{aligned}$$

Since

$$\frac{s-2}{s^2-s-6} = \frac{s-2}{(s-3)(s+2)} = \frac{1}{5}\frac{1}{s-3}+\frac{4}{5}\frac{1}{s+2}$$

So

$$y(t) = \mathcal{L}^{-1}\left\{Y(s)\right\} = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}.$$

■

16.

$$y" + 2y' + 5y = 0, \quad y(0) = 2 \; y'(0) = -1$$

Take the Laplace Transform of the equation

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] + 5Y(s) = 0$$

Plug in the values for initial conditions and simplify, we get

$$\begin{aligned} \left(s^2 + 2s + 5\right) Y(s) &= 2s + 3 \\ \Longrightarrow Y(s) &= \frac{2s+3}{s^2 + 2s + 5} \end{aligned}$$

Since

$$\frac{2s+3}{s^2+2s+5} = \frac{2(s+1)+1}{(s+1)^2 + 2^2} = 2\frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

So

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{Y(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{2s+3}{s^2+2s+5}\right\} \\ &= 2e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t. \end{aligned}$$

■

23.

$$y" + 2y' + y = 4e^{-t}, \quad y(0) = 2, \; y'(0) = -1$$

Take the Laplace Transform of the differential equation, we get

$$\left[s^2 Y(s) - sy(0) - y'(0)\right] + 2\left[sY(s) - y(0)\right] + Y(s) = \frac{4}{s+1}$$

Plug in the initial values and simplify, we get

$$\begin{aligned} \left(s^2 + 2s + 1\right) Y(s) &= \frac{4}{s+1} + 2s + 3 \\ Y(s) &= \frac{2s^2 + 5s + 7}{(s+1)^3} \end{aligned}$$

Reform $Y(s)$ as follows

$$\begin{aligned} Y(s) &= \frac{2s^2+5s+7}{(s+1)^3} \\ &= \frac{2(s+1)^2+(s+1)+4}{(s+1)^3} \\ &= 2\cdot\frac{1}{s+1}+\frac{1}{(s+1)^2}+2\cdot\frac{2!}{(s+1)^3} \end{aligned}$$

So take the inverse Laplace Transform, we get

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{2\cdot\frac{1}{s+1}+\frac{1}{(s+1)^2}+2\cdot\frac{2!}{(s+1)^3}\right\} \\ &= 2e^{-t}+e^{-t}t+2e^{-t}t^2 \end{aligned}$$

■