1. Solve

$$u_t = \alpha^2 u_{xx} + e^{-t} + e^{-2t}\cos(\frac{3\pi x}{L}) \quad 0 < x < L,\ t > 0 \tag{1}$$

with $2 \neq k(\frac{3\pi\alpha}{L})^2$, where k is any integer, and

$$u_x(0,t) = 0 \tag{2}$$
$$u_x(L,t) = 0 \tag{3}$$
$$u(x,0) = 0 \tag{4}$$

2. If

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0 \end{cases} \tag{5}$$

, what are the even and odd parts of $f(x)$? Having found those, on the interval $[-L, L]$, compute their respective Fourier series expansions. What then is the Fourier series for $f(x)$ on the interval $[-L, L]$?

3. Let

$$u_t = \alpha^2 u_{xx} \quad 0 < x < 1,\ t > 0 \tag{6}$$

$$u_x(0,t) = 0 \tag{7}$$
$$u_x(1,t) = 0 \tag{8}$$
$$u(x,0) = x^2. \tag{9}$$

What is $\lim_{t\to\infty} u(x,t)$? Give a brief, physical, explanation of why this makes sense.

4. Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + (\lambda\beta(x) + \gamma(x))\phi = 0 \tag{10}$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + (\lambda r(x) + q(x))\phi = 0 \tag{11}$$

Given $\alpha(x)$, $\beta(x)$, and $\gamma(x)$, what are $p(x)$, $r(x)$, and $q(x)$?

5. Given the problem

$$\epsilon u_t = u_{xx} - u_x \tag{12}$$

$$u(0,t) = 2 \tag{13}$$
$$u(1,t) = 3 \tag{14}$$
$$u(x,0) = 2(1-x) + 3x. \tag{15}$$

6. Suppose $f(x) = f(x+L)$, and f can be differentiated an infinite number of times.

   (a) Show that $f^{(k)}(x) = f^{(k)}(x+L)$, where $f^{(k)}(x)$ is the k-th derivative of f, for any k.

   (b) Given that f has the Fourier series

$$f(x) = c_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{L}) + B_n \sin(\frac{n\pi x}{L}) \tag{16}$$

   on $[-L, L]$, using the facts that on $[-L, L]$

$$|f^{(k)}(x)| \le M_k \tag{17}$$
$$|\sin(\frac{n\pi x}{L})| \le 1 \tag{18}$$
$$|\cos(\frac{n\pi x}{L})| \le 1 \tag{19}$$
$$, \tag{20}$$

   show

$$|A_n| \le 2\left(\frac{L}{\pi n}\right)^k M_k \tag{21}$$
$$|B_n| \le 2\left(\frac{L}{\pi n}\right)^k M_k \tag{22}$$
$$\tag{23}$$

   for all k.