

# Amath 353 Partial Differential Equations Sample Final examination

University of Washington, Applied Mathematics

December 5, 2007

Name:

Registration number:

Use the tables on trigonometric identities attached to the back of this exam. You have 60 minutes. Good luck.

1. Find the integral representation of the solution to Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad x \in (-\infty, \infty), \quad y \in (0, L) \quad (1)$$

subject to the boundary conditions

$$u(x, 0) = e^{-|x|}, \quad u(x, L) = 0, \quad u(x, y) \rightarrow 0 \text{ as } |x| \rightarrow 0 \quad (2)$$

by defining a suitable Fourier Transform from  $x$  to  $k$  space.  $\mathcal{F}\{e^{-|x|}\} = \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}$ .

2. Using the characteristic variables  $\xi = x - ct, \eta = x + ct$ , derive the general solution to the wave equation

$$u_{tt} = c^2 u_{xx}, \quad x \in (-\infty, \infty) \quad (3)$$

3. Calculate the steady-state temperature distribution outside the disk of radius  $a$ ,  $r > a$ , when the temperature distribution on the boundary  $r = 1$  is  $u(r = 1, \theta) = \cos^3 \theta$ .

4. Solve the one dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad x < 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0 \quad (4)$$

using the D'Alembert solution, The boundary condition at  $x = 0$  is  $u(x = 0, t) = 0$ . Draw a space-time diagram indicating the important characteristics. Also draw a diagram showing the evolution of an initially square profile.

5. Derive a weak solution of the conservation law

$$\rho_t + 3\rho^2\rho_x = 0, \quad \rho(x, 0) = \begin{cases} 4, & x < 0 \\ 3, & x > 0 \end{cases} \quad (5)$$

The Rankine-Hugoniot conditions are

$$\frac{dx_s}{dt} = \frac{q(\text{right of } x_s) - q(\text{left of } x_s)}{\rho(\text{right of } x_s) - \rho(\text{left of } x_s)} \quad (6)$$

where  $q(\rho)$  is the flux of the conservation law  $\rho_t + q_x = 0$ . Plot an  $x-t$  diagram showing the important quantities as well as a sequence of snapshots that show the evolution of the initial profile.