Amath 353 Partial Differential Equations Sample midterm questions

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October 26, 2007

Abstract

Questions:

1. State the conditions of the Dirichlet's theorem.

2. What is the value of the Fourier series representation of the function

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 3 \end{cases} \tag{1}$$

   at the point $x = 0$? At the point $x = 7$?

3. Does the function x^3 have a Fourier cosine series in $(-\pi, \pi)$?

4. State Parseval's theorem. It is used to solve what kind of problems?

5. Classify the following equations (parabolic, elliptic, hyperbolic).$x, y \in (-\infty, \infty)$

$$15u_{xx} + u_{xy} - 10u_{yy} = x - 3 \tag{2}$$

$$u_{xx} + xyu_{yy} = 0, \tag{3}$$

   by first constructing a suitable quadratic form and examining the sign of its determinant.

6. Farlow problem 7.4 For the Sturm-Liouville eigenvalue problem

$$y''(x) + \lambda y(x) = 0, \quad y'(0) = 0, \quad y'(L) = 0, \tag{4}$$

   find the eigenvalues and corresponding eigenfunctions (examine all possible values of λ.Then verify that
   (i) There is an infinite number of eigenvalues with a smallest but no largest.
   (ii) The n-th eigenfunction has $n - 1$ zeros.
   (iii) The eigenfunctions are orthogonal

7. Heat equation with convection and lateral surface heat loss. Reduce the parabolic pde

$$u_{xx} + 4u_x - 2u_t + 8u = 0 \tag{5}$$

to the one-dimensional heat equation
Solution

$$u(x,t) = e^{-12t-2x} w(x,t), \quad w_t = \frac{1}{2} w_{xx} \tag{6}$$

8. Solve the non-homogeneous equation

$$u_t = u_{xx} + e^{-2t} \sin 5x, \quad u(0,t) = 1, u(\pi,t) = 0, \quad u(x,0) = 0 \tag{7}$$

Solution
To get rid of the non-homogeneous bcs we introduce the new variable $U(x,t)$ through

$$u(x,t) = \frac{\pi - x}{\pi} + U(x,t) \tag{8}$$

The U satisfies

$$U_t = U_{xx} + e^{-2t} \sin 5x, \quad U(0,t) = 0, U(\pi,t) = 0, \quad U(x,0) = \frac{\pi - x}{\pi} \tag{9}$$

the eigenfunction of the corresponding homogeneous system are $X_n = \sin nx$. Thus with the usual ansatz for the solution of the non-homogeneous equation, we find the solution

$$u(x,t) = \frac{\pi - x}{\pi} + \sum_{n \neq 5} \frac{\pi}{2n} e^{-n^2 t} \sin nx + (\frac{1}{23} e^{-2t} + (\frac{\pi}{10} - \frac{1}{23}) e^{-25t}) \sin 5x \tag{10}$$

9. Given

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases} \tag{11}$$

Sketch the even function f_c of period 4, the odd function f_s of period 4 and the function f_p of period 2 each of which equals $f(x)$ on $(0,2)$. Expand each one in an appropriate Fourier series. Expand f_p in a sine-cosine series and a complex exponential series.
Solution

$$f_s(x) = \frac{2}{\pi}\left(\sin\frac{\pi x}{2} - \sin \pi x + \frac{1}{3}\sin\frac{3\pi x}{3} + ...\right) \tag{12}$$

$$f_c(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{3} - \frac{1}{5}\cos\frac{5\pi x}{5} + ...\right) \tag{13}$$

$$f_p(x) = \frac{1}{2} - \frac{2}{\pi}\sum_{oddn} \frac{1}{n}\sin n\pi x = \frac{1}{2} + \frac{i}{\pi}\sum_{n=-\infty}^{\infty} \frac{1}{n} e^{in\pi x} \tag{14}$$

10. What are the properties characterizing a S-L eigenvalue problem?
(i) There is an infinity of discrete eigenvalues λ_n (they are 'quantized') and an infinity of corresponding eigenfunctions $X_n(x)$.
(ii) Eigenfunctions corresponding to different eigenvalues are orthogonal
(iii) They are complete.

11. Consider the non-homogeneous boundary conditions are imposed on the heat equation $u_t = \alpha^2 u_{xx}$,

$$u(0,t) = k_1 \tag{15}$$

$$u(L,t) = k_2 \tag{16}$$

How is the inhomogeneity eliminated? If you introduce a new function U, state what equation, boundary and initial conditions it satisfies.

To eliminate the non-homogeneity, we introduce a new function $U(x,t)$ through

$$u(x,t) = k_1 + \frac{x}{L}(k_2 - k_1) + U(x,t). \tag{17}$$

Then U satisfies a heat equation with *homogeneous* BCs. However the ICs change.

Given a general PDE

$$3u_{xx} + 4u_{yy} + 5u_{zz} + 2u_x + 2u_y + 2u_z + 100u = 0, \tag{18}$$

eliminate the lower order derivatives.
Solution
Introduce a new function $w(x,y,z)$ through

$$u(x,y,z) = e^{c_1 x + c_2 y + c_3 z} w(x,y,z), \tag{19}$$

substituting into (18) and determining the unknown c_1, c_2, c_3 so that lower derivatives do not occur in the equation for w.

12. Given the heat equation

$$u_t = \alpha^2 u_{xx} - \beta u \tag{20}$$

find the solution up to arbitrary constants.
Solution
First perform the change of vars $u(x,t) = e^{-\beta t} w(x,t)$, to reduce the pde to $w_t = \alpha^2 w_{xx}$. Then the BCs on w will be simple enough so that we can use the table that gives the eigenfunctions and eigenvalues immediately.

13. Given the heat equation

$$u_t = \alpha^2 u_{xx} - \nu u_x \tag{21}$$

and some BCs find the solution up to some arbitrary constants.
Solution
with $u(x,t) = e^{\nu(x-vt)/2\alpha^2} w(x,t)$, the equation reduces to $w_t = \alpha^2 w_{xx}$. Then apply the table to find the solution.

14. Find the Fourier transform of the following function

$$f(x) = \begin{cases} x, & x \in (-1,1) \\ 0, & otherwise \end{cases} \tag{22}$$

Then find an integral representation for the function $f(x)$. What is the value of the integral at $x = \pm 1$?

15. (a) Solve the diffusion equation with convection

$$u_t = ku_{xx} + cu_x, \quad x \in (-\infty, \infty), \quad u(x,0) = f(x) \tag{23}$$

Use the convolution and the shift theorem.
(b) Consider the initial condition to be $\delta(x)$. Sketch the corresponding $u(x,t)$ for various positive times. Comment on the significance of the convection term cu_x.