Amath 353 Partial Differential Equations Highlights Part II

Eleftherios Kirkinis

December 7, 2007

Abstract

Review of solution techniques developed and the corresponding *applied* mathematical constructions.

1 General solution of the wave equation in 1-D

Consider the wave equation in a infinite domain

$$u_{tt} = c^2 u_{xx}, \quad x \in (-\infty, +\infty) \tag{1}$$

We can't solve it like this. We introduce *characteristic* coordinates

$$\xi = x - ct, \quad \eta = x + ct. \tag{2}$$

Substituting into (1) for $u = u(\xi, \eta)$ we obtain

$$u_{\xi\eta} = 0. \tag{3}$$

Integrating twice,

$$u = F(\xi) + G(\eta), \tag{4}$$

where the functions F, G are arbitrary, or

$$u = F(x - ct) + G(x + ct) \tag{5}$$

which is the general solution to the wave equation (1). We also used the sloppy notation of denoting a function of (x, t) and (ξ, η) with the same symbol u.

2 D'Alembert's solution for the wave equation

Now assume we consider the eq. (1) with initial data

$$u_{tt} = c^2 u_{xx}, \quad x \in (-\infty, +\infty) \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \tag{6}$$

How can we use the general solution derived above to incorporate the initial data? We just substitute in (5) and we derive the two relations

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x_0}^{x} g(z) dz + K \tag{7}$$

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^{x} g(z) dz + K \tag{8}$$

Substituting into (5) we obtain the D'Alembert's solution to the wave equation (6)

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)dz \tag{9}$$

after some rearrangement

2.1 The WE in the presence of boundaries: semi-infinite domain

The above general solution and D'Alembert's solution can be used to find the solution to a semi-infinite domain for ex. consider the case

$$u_{tt} = c^2 u_{xx}, \quad x \in (0, +\infty) \quad u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{10}$$

along with the boundary condition $u(0,t) = 0$. The above GS is ok if $x > ct$ but not ok in the region $x < ct$ because the argument of F becomes negative. Thus we need to find an expression that will replace F in the $x < ct$ region. The answer comes from using the boundary condition:

$$u(0,t) = F(-ct) + G(ct) = 0, \quad F(z) = -G(-z), \tag{11}$$

Thus the G.S. for $x < ct$ can be written as

$$u(x,t) = -G(-(x-ct)) + G(x+ct) \tag{12}$$

and we can also write an expression that incorporates the initial conditions. *Understand these type of problems before you come to the exam.*

2.2 The WE in the presence of boundaries: finite domain

If the string is finite, its vibrations are described by

$$u_{tt} = c^2 u_{xx}, \quad x \in (0, L) \quad u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{13}$$

with boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{14}$$

(other types of BC also exist). The solution is obtained with separation of variables

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin k_n x \tag{15}$$

where $\omega_n = ck_n, k_n = \frac{n\pi}{L}$. One then incorporates the initial conditions to find A_n, B_n. What is the geometrical interpretation of the mode

$$u_n(x,t) = \sin \omega_n t \sin k_n x? \tag{16}$$

Fixing x, then u_n behaves as $C_n \sin \omega_n t$. With time the string oscillates with frequency ω_n (radians per second) and the amplitude of oscillation is fixed (at fixed x). At other x the string oscillates with other amplitudes but *the same frequency.* This is the frequency of the musical tone.

Rewriting the mode in terms of $k_n x - \omega_n t$ and $k_n x + \omega_n t$ leads to the usual profile that propagates. (here the profile is sinusoidal). *Understand the geometrical interpretation before you come to the exam.*

3 Elliptic PDEs

Standard examples are the Laplace, Poisson and Helmholtz equations

$$\nabla^2 u = 0, \quad \nabla^2 u = f(x), \quad \nabla^2 u + k^2 u = 0, \tag{17}$$

We briefly talked about the first one, as it represents the *steady-state temperature distribution* in a given domain. Normally, as we did in the homework, you solve these problems with separation of variables.

4 First Order PDEs; The method of characteristics

4.1 Linear 1st order

These are equations of the form

$$a(x,t)u_t + b(x,t)u_x = d(x,t)u, \quad u = u(x,t) \tag{18}$$

and the coefficients a, b, d are known functions of their arguments. Considering the rate of change of $u(x(t), t)$ as measured by a moving observer $x = x(t)$,

$$\frac{d}{dt}w(x(t),t) = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{19}$$

then if the observer moves with velocity b/a then comparing the above equations

$$\frac{dx}{dt} = \frac{b}{a} \tag{20}$$

$$\frac{du}{dt} = \frac{d}{a}u \tag{21}$$

This system is called the characteristic system and the curves $x = x(t)$ (base) characteristic curves. Consider the simple PDE

$$u_t + cu_x = 0 \quad \text{(IC) } u(x,0) = f(x) \tag{22}$$

Labeling a position on the x-axis at t=0 by ξ the characteristic system gives

$$x = ct + \xi, \quad u = F(\xi) \tag{23}$$

where F is an arbitrary function. Thus the general solution to (22) is

$$u(x,t) = F(x - ct). \tag{24}$$

Incorporating the initial data we obtain

$$u(x,t) = f(x - ct). \tag{25}$$

i.e. a translation of the initial profile to the right at speed c. Notes:
The initial data propagate along characteristics. For the above problem, u is constant as we walk along one characteristic but it might change if we jump onto another characteristic.

4.2 Quasi-linear 1st order PDEs

These are equations of the form

$$a(x,t,u)u_t + b(x,t,u)u_x = d(x,t,u), \quad u = u(x,t) \tag{26}$$

and the coefficients a, b, d are known functions of their arguments. The steps to derive the characteristic system are analogous. But now, the phase speed b/a depends on how high the profile is. Characteristics are not parallel anymore and they might intersect. If they do, the solution at the point of intersection becomes multi-valued. There are two important cases of waves that may occur as such: expansion or rarefaction waves (fans) and shock or compression waves.

4.2.1 Rarefaction waves

Consider for example the equation

$$\rho_t + 2\rho\rho_x = 0, \quad \rho(x,0) = \begin{cases} 3, & x < 0 \\ 4, & x > 0 \end{cases} \tag{27}$$

This implies that waves on the negative x axis propagate with velocity $2*3$ thus slower than the waves that emanate from the positive x axis. For each ρ value between 3 and 4 we thus must introduce a fan that takes care of the expansion region. This is given by

$$x = 2\rho t, \quad 3 < \rho < 4 \tag{28}$$

or

$$\rho = \frac{x}{2t}, \quad 6t < x < 8t \tag{29}$$

Thus the solution is

$$\rho(x,t) = \begin{cases} 3, & x < 6t \\ 4, & x > 8t \\ \frac{x}{2t} & 6t < x < 8t \end{cases} \tag{30}$$

4.2.2 Shock waves

Consider for example the equation

$$\rho_t + 2\rho\rho_x = 0, \quad \rho(x,0) = \begin{cases} 4, & x < 0 \\ 3, & x > 0 \end{cases} \tag{31}$$

Now waves emanating from the negative x-axis immediately overtake those from the positive x-axis since the former travel at speed $2*4$ and the latter slower at speed $2*3$. The solution becomes multi-valued and is no longer valid. Instead we expect that there will be a shock wave $x_s(t)$ joining the characteristics from the different regions. To find the position of the shock wave at any time t we consider the Rankine-Hugoniot conditions for the shock velocity

$$\frac{dx_s}{dt} = \frac{q(\text{right of } x_s) - q(\text{left of } x_s)}{\rho(\text{right of } x_s) - \rho(\text{left of } x_s)} \tag{32}$$

where $q(\rho)$ is the flux of the conservation law

$$\rho_t + q_x = 0. \tag{33}$$

For this particular problem, $q = \rho^2$ and the Rankine-Hugoniot conditions become

$$\frac{dx_s}{dt} = 7 \quad x_s(t) = 7t \tag{34}$$

because the shock starts at x=0 and t=0. Thus the solution is

$$\rho(x,t) = \begin{cases} 4, & x < 7t \\ 3, & x > 7t \end{cases} \tag{35}$$

As this is one of the most important topics you will ever learn in PDEs, try to understand these type of problems before you come to the exam.

5 Solution of PDEs in curvilinear coordinates

The equations remain the same but because the boundaries are given in curved domains (circles, ellipses, spheres, cylinders) we need to rewrite the differential operators and notable ∇^2 in these coordinates.

5.1 Laplace's equation in plane-polar coordinates

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{36}$$

With separation of variables the solution is

$$u = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\theta + (C_n r^n + D_n r^{-n}) \sin n\theta \tag{37}$$

We distinguish various cases
(i) The steady-state temperature distribution inside a disk $0 < r < 1$ with the temperature specified on the boundary $u(r = 1, \theta) = g(\theta)$, g given function. In this case we discard the ln and the r^{-n} terms in (37) since they become unbounded at the origin. The left-over should be used to determine the constants A_n, C_n through Fourier series. In the exam you will get one of these problems where $g(r = 1, \theta) = \cos^3\theta$ for example. *Solve this problem before the exam date.*

(ii) Annulus $a < r < b$. Here nothing becomes unbounded and we have to retain all terms and thus determine through FS all constants. To do so we are equipped with the temperature distribution at $r = a$, $u(r = a, \theta) = g_1(\theta)$ and the temperature distribution at $r = b$, $u(r = b, \theta) = g_2(\theta)$, g_1, g_2 known functions.

(iii) The steady-state temperature distribution outside a disk $r > 1$ with the temperature specified on the boundary $u(r = 1, \theta) = g(\theta)$, g given function. In this case we discard the ln and the r^n terms in (37) since they become unbounded at infinity. The left-over should be used to determine the constants B_n, D_n through Fourier series. In the exam you will get one of these problems where $g(r = 1, \theta) = \sin^3\theta$ for example. *Solve this problem before the exam date.*