# Amath 353 Partial Differential Equations Highlights; Part I

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**Abstract**

Review of solution techniques developed and the corresponding *applied* mathematical constructions. Not all subjects to be examined are included here.

## 1 Exam I Review

Exam I will be comprehensive. Everything we covered from hw2 up to and including homework 6. No characteristics, no applications just methodology to solve problems. In particular give special attention to the following homework problems

HW2: Problems 1,2,3,4.
HW3: 1,2,3,6,7
HW4: Problems 1,2,3,5(the Parseval Part), 7,8.
HW5: All but 6.
HW6: All.
Full solution to some of the problems are in the three *Fourier series* hand-outs. The hand out on average values and Parseval should also be reviewed.

Review the accompanying document with possible exam questions. Last year exam is similar to your midterm. If you solve and understand the problems of 2007 midterm you most likely be in the % 70 range. This document is not very well proof read so there might be some typos below.

## 2 Solution methods for homogeneous (no source) heat equation in a finite domain

We are solving

$$u_t = \alpha^2 u_{xx}, \quad x \in (0, L) \tag{1}$$

| $BCs$ | $Dirichlet BCs (settemp)$ | $Neumann BCs (insulated)$ | $Periodic$ |
|---|---|---|---|
| | $X(0) = 0, X(L) = 0$ | $X'(0) = 0, X'(L) = 0$ | $X(-L) = X(L), X'(-L) = X'(L)$ |
| $evalues \lambda_n$ | $(\frac{n\pi}{L})^2$ | $(\frac{n\pi}{L})^2$ | $(\frac{n\pi}{L})^2$ |
| $n =$ | $1, 2...$ | $0, 1, 2, ...$ | $0, 1, 2, 3, ...$ |
| $efns X_n$ | $\sin \frac{n\pi x}{L}$ | $\cos \frac{n\pi x}{L}$ | $\sin \frac{n\pi x}{L} and \cos \frac{n\pi x}{L}$ |

(2)

The solution is given in general by

$$u(x,t) = \sum_{n=0or1}^{\infty} A_n T_n(t) X_n(x), \tag{3}$$

and for the above BCs, $T_n(t) = e^{(\lambda_n \alpha)^2 t}$

**Method**: Separation of variables: $u(x,t) = X(x)T(t)$. Determine $X, T$ by substituting into (8).

Separating variables in an partial differential equation, gives rise to two applied mathematical problems.

- As described in the table above, the function $X(x)$ that describes the spatial variation of the temperature, will satisfy a Sturm-Liouville eigenvalue problem (Lesson 7) of the form

$$X'' + \lambda^2 X = 0, \quad X(0) = 0, \quad X(L) = 0 \tag{4}$$

  (these are Dirichlet BC's) depending on the boundary conditions employed. Nevertheless, a S-L problem is characterized by the following properties:
  (i) There is an infinity of discrete eigenvalues $\lambda_n$ (they are 'quantized') and an infinity of corresponding eigenfunctions $X_n(x)$.
  (ii) Eigenfunctions corresponding to different eigenvalues are orthogonal
  (iii) They are complete.

- In order to satisfy the ICs at $t = 0$, we evaluate the solution (3) at $t = 0$ and equate with $f(x)$ (since the initial conditions given are $u(x, t = 0) = f(x)$, $f(x)$ is a given known function) to obtain

$$f(x) = \sum_{n=0or1}^{\infty} A_n X_n(x), \tag{5}$$

  which is a Fourier series representation (see hand-outs) of the *given* function $f(x)$. Thus, to find the solution, we need to determine the 'integration constants' $A_n$ and these are determined by the usual integrals constructed by calculating average values at both sides

$$A_n = \frac{\int_0^L X_n(x) f(x) dx}{\int_0^L X_n^2(x) dx} \tag{6}$$

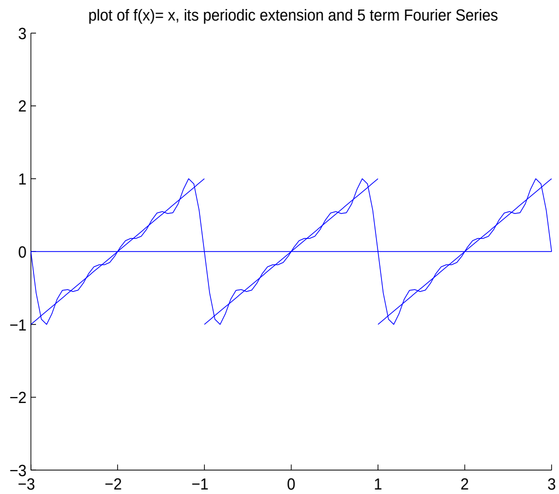


Figure 1: Approximation of $f(x) = x$ by its Fourier Series

Example 1: in the matlab code `fourierseries1.m` we show the graph of $f(x) = x$ on $(-1, 1)$, its periodic extension and the Fourier series

$$f(x) \sim FS = \frac{2}{\pi}(\sin \pi x - \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x + ...) \tag{7}$$

In figure 1 we see that increasing the number of terms in the FS leads to a better approximation of $f(x)$.

Example2: Consider the solution of the heat equation with Neumann (insulated) boundary condition

$$u_t = \alpha^2 u_{xx}, \quad x \in (0, 1/2), \quad u_x(0, t) = u_x(1/2, t) = 0 \tag{8}$$

and initial condition

$$u(x, t = 0) = f(x) = x^2, \quad x \in (0, 1/2). \tag{9}$$

We know that the soln to the PDE is an infinite sum of cosines. This implies that the function to be expanded is $f(x) = x^2$ on $(-1/2, 1/2)$, i.e. expand as an even function. This leads to

$$f(x) \sim FS = \frac{1}{12} - \frac{1}{\pi^2}(\cos 2\pi x - \frac{1}{2^2} \cos 4\pi x + \frac{1}{3^2} \cos 6\pi x + ...) \tag{10}$$

from which we can read off the solution of the PDE. I Figure 2 we show how this solution decays to the value $1/12$ all over the interval $(0, 1/2)$ Why does it not decay to zero? and what is the physical meaning of this behavior?

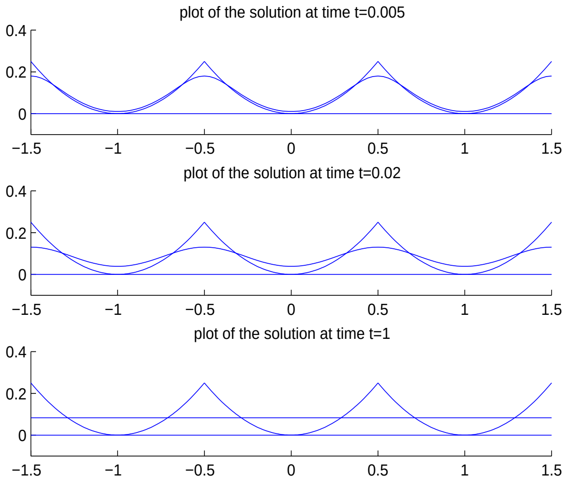


Figure 2: Evolution of the solution to eq. (8) with initial profile $f(x) = x^2$ for the rod in the interval (0,1/2)

# 3 Non-homogeneous Boundary conditions Lesson 6

In general the bc's imposed on $u(x,t)$ are :

$$\alpha_1 u_x(0,t) + \beta_1 u(0,t) = g_1(t) \tag{11}$$

$$\alpha_2 u_x(L,t) + \beta_2 u(L,t) = g_2(t) \tag{12}$$

In the exam we will consider the simpler case

$$u(0,t) = k_1 \tag{13}$$

$$u(L,t) = k_2 \tag{14}$$

To eliminate the non-homogeneity, we introduce a new function $U(x,t)$ through

$$u(x,t) = k_1 + \frac{x}{L}(k_2 - k_1) + U(x,t). \tag{15}$$

Then $U$ satisfies a heat equation with *homogeneous* BCs. However the ICs change.

# 4 Elimination of lower order derivatives in the PDE Lesson 8

Given a general PDE

$$au_{xx} + bu_{yy} + cu_{zz} + du_x + eu_y + fu_z + u = 0, \tag{16}$$

$a, b, c, d, e, f$ given, it is possible to eliminate the lower order derivatives by introducing a new function $w(x, y, z)$ through

$$u(x, y, z) = e^{c_1 x + c_2 y + c_3 z} w(x, y, z), \tag{17}$$

substituting into (16) and determining the unknown $c_1, c_2, c_3$ so that lower derivatives do not occur in the equation for $w$. Special cases from Farlow

- with $u(x, t) = e^{-\beta t} w(x, t)$, the equation

$$u_t = \alpha^2 u_{xx} - \beta u \tag{18}$$

  reduces to $w_t = \alpha^2 w_{xx}$

- with $u(x, t) = e^{\nu(x - vt/2)/2\alpha^2} w(x, t)$, the equation

$$u_t = \alpha^2 u_{xx} - \nu u_x \tag{19}$$

  reduces to $w_t = \alpha^2 w_{xx}$.

# 5 Non-homogeneous PDE's (with source functions or external forcing) Lesson 9(Win 2009 nonexaminable)

The general problem is: Given a heat equation with a source function $g(x, t)$ ($g$ is known)

$$u_t = \alpha^2 u_{xx} + g(x, t) \tag{20}$$

with BCs

$$\alpha_1 u_x(0, t) + \beta_1 u(0, t) = 0 \tag{21}$$

$$\alpha_2 u_x(L, t) + \beta_2 u(L, t) = 0 \tag{22}$$

and ICs $u(x, t = 0) = f(x)$, $f$ known, use an assumed form of the solution

$$u(x, t) = \sum T_n(t) X_n(x). \tag{23}$$

The $X_n$ are known and are the eigenfunctions of the corresponding homogeneous (no $g$) problem with the same BCs. Assuming an expansion for $g$ in the form

$$g(x, t) = \sum g_n(t) X_n(t), \tag{24}$$

where the $g_n$ are known, we need to determine the $T_n(t)$'s. We substitute the assumed solution into the non-homogeneous pde (20) and this leads to an equation for $T_n(t)$ in the form

$$\dot{T}_n(t) + \alpha^2 \lambda_n^2 T_n(t) = g_n(t) \tag{25}$$

# 6 Other applied mathematical techniques

Parseval relates the average of the square of a function over a period with the Fourier coefficients of the series that represents the function.

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2, \tag{26}$$

where $c_n$ are the Fourier coefficients of the *complex* Fourier series representing $f(x)$. The bottom line is that the average total energy, is proportional to the sum of the energies of each individual mode.

# 7 Fourier Transforms highlights

An important by product of the Fourier Transform technique is the calculation of integrals and the integral representation of certain functions. For example in class we showed that the seemingly intractable integral

$$f(x) = \frac{-i}{\pi}\int_{-\infty}^{\infty}\frac{k}{1+k^2}e^{ikx}dx \tag{27}$$

can easily be calculated by resorting to the actual form of $f(x)$ that we used to derive it i.e.

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ -e^{x}, & x < 0 \end{cases} \tag{28}$$

Furthermore, the Fourier Integral Theorem guarantees that at a point of jump of $f(x)$, the integral will converge to the mid-point of the jump, and thus giving us zero in the above problem.

While in Fourier $k$-space, given $F_1(k)\cdot F_2(k)$, where $\mathcal{F}^{-1}\{F_1(k)\} = f_1(x)$ and $\mathcal{F}^{-1}\{F_2(k)\} = f_2(x)$ known functions, what is the inverse Fourier transform of the product $F_1(k)\cdot F_2(k)$? i.e. what is

$$\mathcal{F}^{-1}\{F_1(k)\cdot F_2(k)\}? \tag{29}$$

The answer is the *convolution* of $f_1$ with $f_2$:

$$(f_1 * f_2)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x-u)f_2(u)du \tag{30}$$

Using this property of the FT's we can now solve many PDEs in the infinite domain i.e. all the problems in homework #6.

# 8 Important functions for solving PDEs

These are the Dirac $\delta$ function and the Gaussian or Normal (distribution) function. Consider the latter with the following forms in configuration and Fourier space respectively

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \quad F(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2\sigma^2}{2}} \tag{31}$$

and it is given that the Gaussian function $f$ satisfies the condition

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx = 1 \tag{32}$$

(and thus $F$ satisfies a similar equation). Below we graph $f$ and $F$ as we vary the parameter $\sigma$. This is called the width parameter and measures the distance from the center axis to the inflection point of $f$: Also the Dirac delta function and solution of PDEs as these are

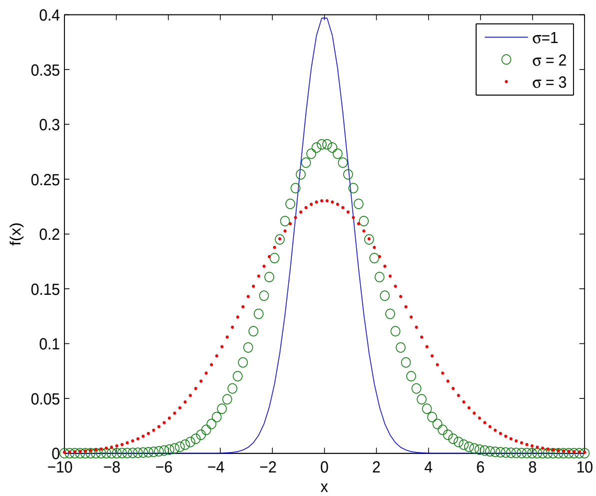


Figure 3: Gaussian function $f(x)$ in (31). Notice that as $\sigma \to 0$, the graph becomes narrower and the amplitude increases

described in the corresponding hw.

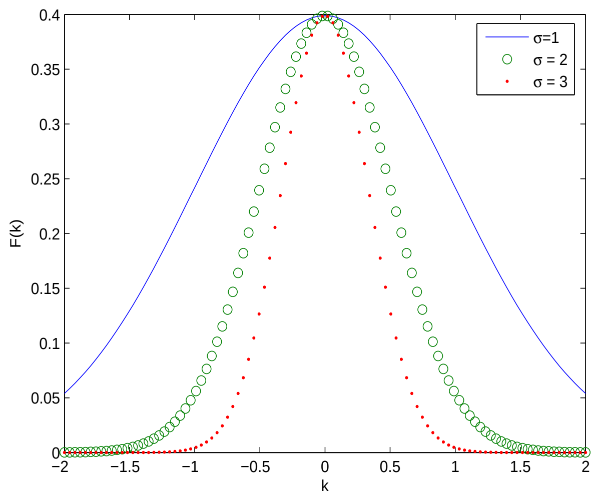


Figure 4: The FT of the Gaussian function $f(x)$ is again a Gaussian $F(k)$ in (31). Notice that as $\sigma \to 0$, the graph becomes wider and the amplitude remains constant. The former happens because the width-parameter for $F$ is $1/\sigma$.