Amath 353 Partial Differential Equations Mid-term examination

University of Washington, Applied Mathematics

Monday, November 5, 2007

Name:
Registration number:
Use the tables on trigonometric identities and boundary value problems attached to the back of this exam. You have 50 minutes. Good luck.

1. (a) Classify the following differential equation by deriving a suitable quadratic form

$$xu_{xx} + yu_{yy} = 0, \tag{1}$$

where $u = u(x, y)$ and $x, y \in (-\infty, \infty)$.
Solution
The quadratic form is

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \tag{2}$$

Thus elliptic det $A > 0$ when xy¿0 i.e. in the first and third quadrants, hyperbolic when det $A < 0$ when xy¡0 in the second and fourth quadrants parabolic elsewhere.

(b) With a suitable functional transformation eliminate the right hand side term from the following PDE

$$u_t + cu_x = u, \quad u = u(x, t), \quad c = \text{constant}. \tag{3}$$

Solution
Let $u(x, t) = e^{c_1 t + c_2 x} w(x, t)$ with the constants to be determined. Substitute into the PDE to obtain

$$w_t + cw_x = (1 - cc_2 - c_1)w \tag{4}$$

various choices are possible. For ex. $c_2 = 0, c_1 = 1$ leads to $u = e^t w$ etc.

2. Use Parseval's Theorem to find the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... \tag{5}$$

using the Fourier series representation

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{\ell} + \frac{1}{3}\sin\frac{3\pi x}{\ell} + + \frac{1}{5}\sin\frac{5\pi x}{\ell} + ...\right) \tag{6}$$

of the function

$$f(x) = \begin{cases} -1, & -\ell < x < 0 \\ 1, & 0 < x < \ell. \end{cases} \tag{7}$$

Solution
Parseval: integrate the square of the function over a period

$$\frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)^2 dx = 1 \tag{8}$$

By Parseval this is equal to the $\frac{1}{2}\sum_{n=1}^{\infty} b_n^2$ so

$$1 = \frac{1}{2}\frac{4^2}{\pi^2}(1 + \frac{1}{3^2} + ...) \tag{9}$$

which leads to the value of $\pi^2/8$ for the series.

3. Consider derivative boundary conditions of the form

$$u_x(0,t) = 0, \quad u_x(L,t) = 0. \tag{10}$$

for the heat equation

$$u_t = ku_{xx}, \quad u(x,0) = \sin^4 \frac{\pi x}{L} \tag{11}$$

(a) Find an explicit expression for $u(x,t)$. You might want to use the table on the trigonometric identities.
(b) What is the limit of $u(x,t)$ as $t \to \infty$? If u is the temperature on a finite rod in $(0,L)$ what is the physical significance of the result?
Solution
We know the solution has the form

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt} \cos \frac{n\pi x}{L} \tag{12}$$

To accommodate the IC's, at $t = 0$ we rewrite the ICs as

$$\sin^4 \frac{\pi x}{L} = \frac{3}{8} - \frac{1}{2}\cos 2\frac{\pi x}{L} + \frac{1}{8}\cos 4\frac{\pi x}{L}. \tag{13}$$

This implies that $A_0 = 3/8, A_2 = 1/2, A_4 = 1/8$ all other A's are zero. The solution is

$$u(x,t) = \frac{3}{8} - \frac{1}{2}e^{-(\frac{2\pi}{L})^2 kt} \cos 2\frac{\pi x}{L} + \frac{1}{8}e^{-(\frac{4\pi}{L})^2 kt} \cos 4\frac{\pi x}{L} \tag{14}$$

Note that

$$A_0 = \frac{1}{L}\int_0^L f(x)dx \tag{15}$$

This means that as $t \to 0$ u approaches the average value of the initial data.

$$\lim_{t\to\infty} = \frac{1}{L}\int_0^L f(x)dx \tag{16}$$

and the energy injected through the initial conditions spreads out evenly over the length of the rod. This happens because of the insulated ends.

4. (a) Give the definition of a Sturm-Liouville eigenvalue problem. Describe at least three properties of such a problem.
(b) Consider the eigenvalue problem

$$X''(x) + \lambda^2 X(x) = 0, \quad x \in (0,1), \quad X(0) = 0, X'(1) + X(1) = 0. \tag{17}$$

Find the eigenvalues and eigenfunctions. Show that the eigenvalues satisfy one of the properties you defined in section (a).

5. Solve the heat equation with convection $u_t = \alpha^2 u_{xx} + cu_x$ in the infinite domain $x \in (-\infty, \infty)$ with initial conditions $u(x, t = 0) = f(x)$ (a known function) and implied boundary conditions $u \to 0$ as $x \to \pm\infty$. (Solution in the solutions of hw6)