Amath 353 Partial Differential Equations Mid-term examination

University of Washington, Applied Mathematics

Monday, November 5, 2007

Name:
Registration number:
Use the tables on trigonometric identities and boundary value problems attached to the back of this exam. You have 50 minutes. Good luck.

1. (a) Classify the following differential equation by deriving a suitable quadratic form

$$xu_{xx} + yu_{yy} = 0, \tag{1}$$

where $u = u(x, y)$ and $x, y \in (-\infty, \infty)$.

(b) With a suitable functional transformation eliminate the right hand side term from the following PDE

$$u_t + cu_x = u, \quad u = u(x, t), \quad c = \text{constant}. \tag{2}$$

2. Use Parseval's Theorem to find the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + ... \tag{3}$$

using the Fourier series representation

$$f(x) = \frac{4}{\pi}\left(\sin\frac{\pi x}{\ell} + \frac{1}{3}\sin\frac{3\pi x}{\ell} + +\frac{1}{5}\sin\frac{5\pi x}{\ell} + ...\right) \tag{4}$$

of the function

$$f(x) = \begin{cases} -1, & -\ell < x < 0 \\ 1, & 0 < x < \ell. \end{cases} \tag{5}$$

What is the value of the Fourier series representation at $x = -\ell/2$? What is its value at $x = -3\ell/2$? Why?

3. Consider derivative boundary conditions of the form

$$u_x(0,t) = 0, \quad u_x(L,t) = 0. \tag{6}$$

for the heat equation

$$u_t = ku_{xx}, \quad u(x,0) = \sin^4 \frac{\pi x}{L} \tag{7}$$

(a) Find an explicit expression for $u(x,t)$. You might want to use the table on the trigonometric identities.
(b) What is the limit of $u(x,t)$ as $t \to \infty$? If u is the temperature on a finite rod in $(0,L)$ what is the physical significance of the result?

4. (a) Give the definition of a Sturm-Liouville eigenvalue problem. Describe at least three properties of such a problem.
(b) Consider the eigenvalue problem

$$X''(x) + \lambda^2 X(x) = 0, \quad x \in (0,1), \quad X(0) = 0, X(1) = 0. \tag{8}$$

Find the eigenvalues and eigenfunctions. Show that the eigenvalues satisfy one of the properties you defined in part (a).

5. Solve the heat equation with convection $u_t = \alpha^2 u_{xx} + c u_x$ in the infinite domain $x \in (-\infty, \infty)$ with initial conditions $u(x, t = 0) = f(x)$ (a known function) and implied boundary conditions $u \to 0$ as $x \to \pm\infty$.