3 Hamiltonian Dynamics and Classical Mechanics

3.1 A dynamical system look of harmonic oscillation

Harmonic oscillator is one of the most important solvable models in classical Newtonian mechanics:

$$m\frac{d^2x}{dt^2} = -kx - \eta\frac{dx}{dt}. \tag{50}$$

The equation can be re-written as a pair of linear, first-order autonomous ordinary differential equations with constant coefficients:

$$\frac{dx}{dt} = y, \tag{51a}$$
$$\frac{dy}{dt} = -\frac{k}{m}x - \frac{\eta}{m}y. \tag{51b}$$

The eigenvalue and eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{pmatrix} \tag{52}$$

contains all the essential information about the dynamical system. In particular, its two eigenvalues are

$$r_{1,2} = \frac{-\eta \pm \sqrt{\eta^2 - 4mk}}{2m}. \tag{53}$$

So we see that that if $\eta \neq 0$ (η has to be positive from the physical requirement), then the fixed point $x = y = 0$ is stable. However, it can be either a node, when both $r_{1,2}$ are real negative, or a focus, when $r_{1,2}$ is a pare of complex conjugates. The condition for the complex eigenvalues is

$$\eta^2 < 4mk. \tag{54}$$

This is the traditional approach to the analysis of a harmonic oscillator.

Now we consider a very different approach, one in which we view the dynamics as a ”flow on a circle”. To do that, we first transform the system of differential equations in Eq (51) into polar coordinates:

$$r^2 = kx^2 + my^2, \quad \tan\theta = \sqrt{\frac{m}{k}}\frac{y}{x}. \tag{55}$$

Recall that

$$\frac{d}{dt}\tan\theta = \sec^2\theta\frac{d\theta}{dt}, \quad 1+\tan^2\theta = \sec^2\theta,$$

and,

$$d\theta = \frac{\sqrt{km}(xdy - ydx)}{kx^2+my^2},$$

we have

$$\frac{d}{dt}r^2 = -4ar^2\sin^2\theta \tag{56a}$$

$$\frac{d}{dt}\theta = -\omega - a\sin 2\theta \tag{56b}$$

in which

$$\omega = \sqrt{\frac{k}{m}}, \quad a = \frac{\eta}{2m}. \tag{57}$$

We note that Eq. (56b) is independent of (56a). Now, immediately from what we know about the flow on a circle: $a = \omega$ is a bifurcation point; that is $\eta^2 = 4km$! Furthermore, we know that the period of the rotation is $T = \frac{2\pi}{\sqrt{\omega^2-a^2}}$, which corresponds to the angular velocity of

$$\sqrt{\omega^2 - a^2} = \sqrt{\frac{k}{m} - \frac{\eta^2}{4m^2}} = \frac{\sqrt{4mk-\eta^2}}{2m}!$$

This is precisely the imaginary part of the eigenvalue, when there they are complex.

We can further ask the following two questions: What are the limit, in the long time:

$$\lambda = \lim_{t\to\infty}\frac{\ln r(t)}{t}, \quad \text{and} \quad \alpha = \lim_{t\to\infty}\frac{\theta(t)}{t}. \tag{58}$$

They are called *Lyapunov exponent* and *rotation number*.

To compute the Lyapunov exponent λ, we integrate the Eq. (56a):

$$2\frac{d\ln t}{dt} = -4a\sin^2\theta,$$

$$\ln r = -2a\int_0^t \sin^2\theta(t)dt$$

$$\lambda = -2a\lim_{t\to\infty}\frac{1}{t}\int_0^t \sin^2\theta(t)dt.$$

If $\omega < a$, $\theta(t) \to \theta^*$, a stable fixed point $\sim (2\theta^*) = -\omega/a$. Therefore

$$\sim^2 \theta^* = \frac{1}{2}\left(1 - \sqrt{1 - \left(\frac{\omega}{a}\right)^2}\right). \tag{59}$$

This,

$$\lambda_{\omega<a} = -2a\sin^2\theta^* = -\frac{\eta - \sqrt{\eta^2 - 4km}}{2m}. \tag{60}$$

If $\omega > a$, then

$$\begin{aligned}
\lambda_{\omega>a} &= -2a \lim_{t\to\infty} \frac{1}{t}\int_0^t \sin^2\theta(t)dt \\
&= -\frac{2a}{T}\int_0^{2\pi} \frac{\sin^2\theta}{\omega + a\sin 2\theta} d\theta \\
&= -\frac{a}{T}\int_0^{2\pi} \frac{1-\cos\phi}{\omega + a\sin\phi} d\phi \\
&= -\frac{a}{T}\left[\frac{2\pi}{\sqrt{\omega^2 - a^2}} - \int_0^{2\pi} \frac{\cos\phi}{\omega + a\sin\phi} d\phi\right] \\
&= -a.
\end{aligned} \tag{61}$$

Therefore, we have

$$\lambda = \begin{cases} -\frac{\eta - \sqrt{\eta^2 - 4km}}{2m} & \omega \le a \\ -a & \omega \ge a. \end{cases} \tag{62}$$

In summary, the λ is the larger real part of the eigenvalues, and α is zero for overdamped oscillation and the imaginary part of the eigenvalues for underdamped oscillation.

3.2 Point transformation and Lagrange equation

Let us consider the point transformation, i.e., the transformation in the coordinate space:

$$Q_i = Q_i(q_j, t), \text{ or } q_j = q_j(Q_i, t). \tag{63}$$

One can show that the Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{64}$$

is invarant under the transformation! To see this, consider the function

$$L\left(q_j, \dot{q}, t\right), \tag{65}$$