Contents

1 Introduction and Review

1.1 A Brief History of Dynamics

1637	Descartes	Analytical geometry
1666	Newton	Calculus, the law of gravity, and the equation of motion
1700s		Flowering of calculus and classical mechanics
1800s		Mathematical studies of planetary motion and many other physical phenomena
1814	Laplace	Causal determinism: "The movements of the greatest bodies of the universe and the tiniest atoms" all follow the equation
1890s	Poincaré	Geometric approach to dynamics and celestial mechanics
1920-1950		Nonlinear oscillations in physics and engineering, invention of radio, radar, laser
1920-1960	Birkhoff Kolmogorov Arnol'd Moser	Complex behavior in Hamiltonian mechanics
1960s	Lorenz Smale	Strange attractor in simple model of convection Topological theory of nonlinear dynamics
1970s	Ruelle & Takens May Feigenbaum	Turbulence and chaos Chaos in simple ecological model: the logistic map Universality and renormalization, connection between chaos and phase transitions
		Experimental studies of chaos
	Winfree Mandelbrot	Nonlinear oscillation in biology Fractal geometry
1980s		Widespread interest in chaos, fractals, oscillations, and their applications
1987		James Gleick published his national bestseller "*Chaos: Making A New Science*"

1.2 Ordinary Differential Equations (ODEs)

1.2.1 The state of a system changing with time

Characterized by *dependent variablbles* that are functions of time.

Instantaneous rate: how the variables change with time.

Relation between the rate and the state of the system.

1.2.2 The origins of ODEs

Mechanics: the equation of motion

Chemistry: the radioactivity decay

Biology: the population growth

1.2.3 Classification of ODEs

ODE or not? (Partial differential equations, integral equations, difference equations/maps, delay differential equations.)

The order of an equation? (The number of initial conditions, infinite dimensional systems.)

Linear or nonlinear?

Homogeneous or inhomogeneous?

Constant coefficient or not?

$$ay''' + by'' + cy' + dy = 0;$$

$$\frac{du}{dt} - au^2 + v(t) = 0;$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y + \sin x = 0;$$

$$xx'' + x^2x' + x = 0;$$

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,y) + \sin t, \\ \dot{y} & = & g(x,y) - \cos t; \end{array} \right.$$

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} + \rho\frac{\partial \rho}{\partial x} + e^{\mu\rho}.$$

1.3 Solving ODEs: Two Examples

1.3.1 Linear first-order, inhomogeneous equation

$$\frac{dx}{dt} = ax + b, \quad (a \neq 0) \tag{1}$$

(i) Standard method.

"The general solution to a linear, inhomogeneous ordinary differential equation is the sum of the general solution to the corresponding homogeneous equation and a particular solution to the inhomogeneous equation."

The homogeneous solution is

$$\frac{dx}{dt} = ax, \tag{2}$$

Which has a constant coefficent. So one tries the exponential function $x = e^{rt}$ and obtains the characteristic equation

$$r = a.$$

Hence, the general solution to the homogeneous equation Eq. (2) is Ce^{at}. By inspection, we have a solution to the inhomogeneous equation $-b/a$. Hence the general solution to the Eq. (1) is

$$\boxed{x(t) = Ce^{at} - \frac{b}{a}}. \tag{3}$$

(ii) Separation of variables. One writes the equation into

$$\frac{dx}{ax + b} = dt. \tag{4}$$

Then we have

$$\int \frac{dx}{ax+b} = t + C',$$
$$\frac{1}{a}\log(ax+b) = t + C',$$
$$x = \frac{e^{a(t+C')} - b}{a},$$
$$\boxed{x = Ce^{at} - \frac{b}{a}},$$

where $C = ae^{aC'}$.

(iii) Integration factor. One first writes the equation into

$$dx - (ax + b)dt = 0, \tag{5}$$

that fits the genearl form of

$$f(x,t)dx + g(x,y)dt = 0. \tag{6}$$

One then check that whether

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial g(x,t)}{\partial x}? \tag{7}$$

This is not true for Eq. (5). Hence, it requires an intgration factor e^{-at}:

$$\begin{aligned}
&e^{-at} d\left(x + \frac{b}{a}\right) + \left(x + \frac{b}{a}\right)(-a)e^{-at}dt = 0, \\
&e^{-at} d\left(x + \frac{b}{a}\right) + \left(x + \frac{b}{a}\right) d\left(e^{-at}\right) = 0, \\
&d\left(e^{-at}\left(x + \frac{b}{a}\right)\right) = 0, \\
&e^{-at}\left(x + \frac{b}{a}\right) = C, \\
&\boxed{x = Ce^{at} - \frac{b}{a}}.
\end{aligned}$$

1.3.2 Nonlinear first-order equation

$$\frac{dx}{dt} = ax - bx^2, \quad (b > 0) \tag{8}$$

There is no standard method for nonlinear ODEs.

Separation of variables. One writes the equation into

$$\frac{dx}{ax - bx^2} = dt. \tag{9}$$

Then we have, by the method of partial fractions

$$\begin{aligned}
&\int \frac{dx}{ax - bx^2} = t + C, \\
&\frac{1}{a}\int \left(\frac{1}{x} + \frac{b}{a - bx}\right) dx = t + C, \\
&\log(x) - \log(a - bx) = a(t + C), \\
&\frac{x}{a - bx} = e^{a(t+C)} \\
&\boxed{x = \frac{ae^{a(t+C)}}{1 + be^{a(t+C)}}}.
\end{aligned}$$

If one is solving an *initial value problem*, then one has $x(0) = x_0$. Substitute this into above we have

$$x_0 = \frac{ae^{aC}}{1 + be^{aC}},$$

which yields $e^{aC} = \frac{x_0}{a - bx_0}$. Therefore, the solution to the initial value problem

$$\begin{cases} \frac{dx}{dt} = ax - bx^2, \quad (b > 0) \\ x(0) = x_0, \end{cases} \tag{10}$$

is

$$\boxed{x = \frac{ax_0e^{at}}{a - bx_0 + bx_0e^{at}}}. \tag{11}$$

1.4 A Geometric Perspective of Analyzing ODEs

1.4.1 ODEs, vector fields, and trajectories

Simple root and fixed point.

How do we find the time for a system to relax back to its stable fixed point? To do this, let us consider what happens near the fixed point. So close to it, we can take Newton's view and assume a linear function at the root. This is the first order approximation of the function

$$f(x) \approx f(x^*) + f'(x^*)(x - x^*) = f'(x^*)(x - x^*). \tag{12}$$

If we move the coordinate system to $u = x - x^*$, then we have a simpler linear differential equation

$$\frac{du}{dt} = -au, \quad \text{where} \quad a = -f'(x^*). \tag{13}$$

The solution to this is

$$u(t) = u(0)e^{-at}.$$

Slower approaching to fixed point

$$\frac{dx}{dt} = -x^3, \quad x(0) = x_o. \tag{14}$$

$$x(t) = \frac{x_o}{\sqrt{1 + 2x_o^2 t}}, \tag{15}$$

this function approaches to zero much slower than the exponential $\sim t^{-1/2}$. It approaches to zero *algebraically*. Sometime called *power law*.

Faster approaching to fixed point

$$\frac{dx}{dt} = -\sqrt[3]{x}, \quad x(0) = x_o. \tag{16}$$

$$x(t) = \left(x_o^{2/3} - \frac{2t}{3}\right)^{3/2}. \tag{17}$$

This function approaches to zero in finite time. In fact, the time is

$$t = \frac{3}{2}x_o^{2/3}. \tag{18}$$

It is still slower than a constant velocity; but it is much faster than exponential approaching.

Note also, since $\sqrt[3]{x}$ is not a "good function", the derivative at $x = 0$ is infinite, the uniqueness of the solution is no longer for sure. The approaching to stable fixed point in finite time has an interesting conseqeunce for unstable fixed point:

$$\frac{dx}{dt} = \sqrt[3]{x}, \quad x(0) = 0. \tag{19}$$

We have

$$\int x^{-1/3}dx = \int dt,$$
$$\frac{3}{2}x^{2/3} = t + C,$$
$$\frac{3}{2}x^{2/3} = t \text{ with initial condition },$$
$$x(t) = \left(\frac{2}{3}t\right)^{3/2}!$$

1.4.2 The global perspective

These are two different perspectives of dynamics, one local, and one global. Ever since the time of Newton, it has been known that the time evolution of a mechanical system can be described by a solution $x(t)$ to a systems of ordinary differential equations provided an initial condition is given: $x(0) = y$. The mathematical space for the dynamics is known as the *phase space*. The dynamical systems perspective were developed much later. It suggests that one should look at the evolution of the whole phase space of initial conditions, as a flow, not only that of a specific initial value y, under the action of an appropriate vector field, or transformations F_t, called a flow in view of the natural analogy with fluids. Therefore, a solution $x(t)$ with an initial condition y can be written as $F_t(y)$. Among early explicit manifestations of this approach was the celebrated Poincar recurrence theorem, whose statement concerns only almost all initial conditions and has nothing to say about a specific one. This is why we call it a global perspective.

2 Two-Dimensional Flow and Nonlinear Planar System

2.1 Two-dimensional linear ODEs with constant coefficients

We show that second-order ODE

$$ay'' + by' + cy = 0 \tag{20}$$

and two-dimensional system of ODEs:

$$\begin{cases} x' &= \alpha x + \beta y, \\ y' &= \gamma x + \delta y. \end{cases} \tag{21}$$

can be transformed between each other. It is straight forward to transform Eq. (20) to the form of Eq. (21).

$$\begin{cases} x' &= -\frac{b}{a}x - \frac{c}{a}y, \\ y' &= x. \end{cases} \tag{22}$$

The converse is a little more challenging. Note if both β and γ are zero, then Eq. (21) really is not a two-dimensional system, but two independent one-dimensional equations. That is not very interesting. Assuming $\gamma \neq 0$, we differentiate the Eq. (21b) once, and have

$$\begin{aligned} \gamma x &= y' - \delta y, \\ \gamma x' &= y'' - \delta y', \\ x' - \alpha x &= \beta y \end{aligned}$$

from which we should be able to eliminate x and x' to obtain an equation only in y,

$$\frac{y'' - \delta y'}{\gamma} - \alpha \frac{y' - \delta y}{\gamma} = \beta y. \tag{23}$$

That is,

$$y'' - (\delta + \alpha)y' + (\alpha\delta - \beta\gamma)y = 0. \tag{24}$$

2.2 Eigenvalues and the characteristic polynomial

The characteristic polynomial for Eq. (20) is obtained by assuming

$$y(t) = e^{rt}. \tag{25}$$

Substituting this into Eq. (20), we have

$$\boxed{r^2 + \frac{b}{a}r + \frac{c}{a} = 0}. \tag{26}$$

If we look at the system of ODEs in Eq. (21), it can be written in a very compact matrix form:

$$\frac{d}{dt}\mathbf{u} = \mathbf{A}\mathbf{u}, \tag{27}$$

where $\mathbf{u}$ is a vector and $\mathbf{A}$ is a matrix

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{28}$$

The eigenvalues of $\mathbf{A}$, λ, satisfies the equation

$$\begin{vmatrix} \alpha - \lambda & \beta \\ \gamma & \delta - \lambda \end{vmatrix} = 0. \tag{29}$$

That is,

$$(\alpha - \lambda)(\delta - \lambda) - \beta\gamma = \lambda^2 - (\alpha + \delta)\lambda + (\alpha\delta - \beta\gamma) = 0. \tag{30}$$

We note that

$$\boxed{tr\mathbf{A} = \alpha + \delta, \quad det\mathbf{A} = \alpha\delta - \beta\gamma}. \tag{31}$$

2.3 Why is e^{rt} so unique? What has it to do with eigenvalue problem?

The connection between guessing exponential function e^{rt} as a solution to linear ODEs with constant coefficients, Eq. (20), and eigenvalues of the matrix $\mathbf{A}$ in Eq. (21) is indeed very real. In fact, if we identify

$$\left(\frac{d}{dt}\right) u(t) \tag{32}$$

as a linear transformation of $u(t)$. Then we have an eigenvalue problem: Finding function $u(t)$ such that

$$\left(\frac{d}{dt}\right) u_r(x) = r u_r(x), \tag{33}$$

in which r is called an eigenvalue and $u_r(x)$ is the corresponding “eigenvector”, or eigenfunction. It is easy to see that e^{rt} is the eigenfunction is e^{rt} since

$$\left(\frac{d}{dt}\right) e^{rt} = re^{rt}! \tag{34}$$

However, we also note that e^{rt} is not the eigenfunction for derivative with non-constant coefficient.

$$\left(e^x \frac{d}{dx}\right) u_r(x) = r u_r(x) \tag{35}$$

is an eigenvalue problem for the linear transformation (*linear differential operator*) $e^x \frac{d}{dx}$. We can easily verify that e^{rt} is no longer an eigen function. Rather we have

$$u_r(x) = e^{-re^{-x}}! \tag{36}$$

What will be the eigen values and eigenfunctions for this linear differential operator

$$\left((1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right) y(x)? \tag{37}$$

The answer to this question has to do with a class of *special functions* called Legendre functions. You will hear more about this in amath 403. And the eigenfunctions associated with this linear differential operator

$$\left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + x^2\right) y(x) \tag{38}$$

is called Bessel functions.

2.4 Eigenvectors and planar phase portrait

Why is the eigenvalue problem important to the ODEs?

Let us now consider a linear two-dimensional system, often also called a planar system, in its vector/matrix form

$$\frac{d}{dt}\mathbf{u} = \mathbf{A}\mathbf{u}. \tag{39}$$

Let us assume that matrix $\mathbf{A}$ has eigenvalues λ_1 and λ_2, with corresponding, linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Then we can express unknown function

$$\mathbf{u}(t) = a_1(t)\mathbf{u}_1 + a_2(t)\mathbf{u}_2, \tag{40}$$

and the corresponding differential equation (39)

$$\frac{da_1(t)}{dt}\mathbf{u}_1 + \frac{da_2(t)}{dt}\mathbf{u}_2 = \mathbf{A}\left(a_1(t)\mathbf{u}_1 + a_2(t)\mathbf{u}_2\right) \tag{41}$$

$$= \lambda_1 a_1(t)\mathbf{u}_1 + \lambda_2 a_2(t)\mathbf{u}_2. \tag{42}$$

Therefore, in the $\mathbf{u}_1$, $\mathbf{u}_2$ coordinate system, we have the differential equation (39) in its simple form

$$\left\{ \begin{array}{rcl} \frac{da_1(t)}{dt} & = & \lambda_1 a_1(t), \\ \frac{da_2(t)}{dt} & = & \lambda_2 a_2(t). \end{array} \right. \tag{43}$$

The two-dimensional system is separable and independent!

The solution to the two equations in (43) are

$$\left\{ \begin{array}{rcl} a_1(t) & = & a_1(0)e^{\lambda_1 t} \\ a_2(t) & = & a_2(0)e^{\lambda_2 t}. \end{array} \right. \tag{44}$$

If one use a_1 and a_2 as the abscissa and ordinate for the planar phase portrait, then we have several different pictures depending on the signs of the λ_1 and λ_2. See textbook P. 127.

2.5 Another look of the problem: matrix exponential

The solution to the differential equation (39) can be formally written as

$$\boxed{\mathbf{u}(t) = e^{\mathbf{A}t}\mathbf{u}(0)} \tag{45}$$

in which the *matrix exponential*

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k. \tag{46}$$

We note that

$$e^{\mathbf{A}0} = \mathbf{I} \tag{47}$$

is the identity matrix. Furthermore,

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}. \tag{48}$$

When matrix $\mathbf{A}$ is diagonalized, so is $e^{\mathbf{A}t}$:

$$\mathbf{A} \longrightarrow \mathbf{\Lambda} = \left(\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right): \quad e^{\mathbf{A}t} \longrightarrow e^{\mathbf{\Lambda}t} = \left(\begin{array}{cc} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{array} \right). \tag{49}$$

3 Hamiltonian Dynamics and Classical Mechanics

3.1 Point transformation and Lagrange equation

Let us consider the point transformation, i.e., the transformation in the coordinate space:

$$Q_i = Q_i(q_j, t), \text{ or } q_j = q_j(Q_i, t). \tag{50}$$

One can show that the Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{51}$$

is invarant under the transformation! To see this, consider the function

$$L\left(q_j, \dot{q}, t\right), \tag{52}$$

in which

$$\left\{\begin{array}{lcl} q_j &=& q_j(Q_i, t), \\ \dot{q}_j &=& \sum_\ell \frac{\partial q_j}{\partial Q_\ell}(Q_i, t)\dot{Q}_\ell + \frac{\partial q_j}{\partial t}(Q_i, t). \end{array}\right. \tag{53}$$

By chain rule:

$$\frac{\partial \dot{q}_j}{\partial Q_i} = \sum_\ell \frac{\partial^2 q_j}{\partial Q_\ell \partial Q_i}\dot{Q}_\ell + \frac{\partial^2 q_j}{\partial t \partial Q_i} = \frac{d}{dt}\left(\frac{\partial q_j}{\partial Q_i}\right). \tag{54}$$

Therefore, as a function of $Q_i, \dot{Q}_i$ and t, we have for L:

$$\begin{aligned} \frac{\partial L}{\partial Q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) &= \sum_j \left\{\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial Q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial Q_i}\right)\right\} \\ &= \sum_j \left\{\frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial Q_i} - \frac{\partial q_j}{\partial Q_i}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right\} \\ &= \sum_j \frac{\partial q_j}{\partial Q_i}\left\{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right\} = 0. \end{aligned}$$

Therefore, the Lagrangian under the point transformation $Q_i = Q_i(q_j, t)$ is just a simple variable substitution.

3.2 The canonical tranaformation and Hamiltonian equation

We have demonstrated that Lagrange equation is invariant under point transformation. It can also be shown that it is canonical:

$$\begin{cases} q = q(Q,t), \;\; \dot{q} = \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial t}, \\ p = P\left(\frac{\partial q}{\partial Q}\right)^{-1}. \end{cases} \tag{55}$$

Thus, we have,

$$\begin{aligned}
\widetilde{H}(Q,P,t) &= \widetilde{L}(Q,\dot{Q},t) + \sum P\dot{Q} \\
&= L\left(q(Q,t), \frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial t}, t\right) + \sum P\dot{Q} \\
&= H\left(q(Q,t), P\left(\frac{\partial q}{\partial Q}\right)^{-1}, t\right) - \sum P\left(\frac{\partial q}{\partial Q}\right)^{-1}\left(\frac{\partial q}{\partial Q}\dot{Q} + \frac{\partial q}{\partial t}\right) + \sum P\dot{Q} \\
&= H\left(q(Q,t), P\left(\frac{\partial q}{\partial Q}\right)^{-1}, t\right) - P\frac{\left(\frac{\partial q}{\partial t}\right)}{\left(\frac{\partial q}{\partial Q}\right)}; \\
\frac{\partial \widetilde{H}}{\partial Q} &= \\
&= -\dot{P}; \\
\frac{\partial \widetilde{H}}{\partial P} &= \\
&= \dot{Q}.
\end{aligned}$$

If the transformation is time independent, $q = q(Q)$, then we have direct substitution

$$\widetilde{H}(Q,P,t) = H\left(q(Q), P(\partial q/\partial Q)^{-1}, t\right). \tag{56}$$

Canonical transformation is a broad class of transformation than the point transformation. Its general form is

$$\begin{cases} q = q(Q,P,t) \\ p = p(Q,P,t) \end{cases} \tag{57}$$

It is important to notice that in general $\widetilde{H}(Q,P,t)$ is not just a simple varaible substitution from $H(q,p,t)$, but rather some complicated from. It is also important to mention the relationship between Lagrangian and Hamiltonian:

$$H = L + \sum p\dot{q}; \text{ or } L = H - \sum p\dot{q} \tag{58}$$

there are three independent variables at the right sides of the equations: p, q, and $\dot{q}$, but there are only two explicit on the left sides:

$$\frac{\partial H}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + p = -p + p = 0,\ \frac{\partial H}{\partial p} = \dot{q},\ \frac{\partial H}{\partial q} = -\dot{p}, \tag{59}$$

$$\frac{\partial L}{\partial p} = \frac{\partial H}{\partial p} - \dot{q} = \dot{q} - \dot{q} = 0,\ \frac{\partial L}{\partial q} = -\dot{p},\ \frac{\partial L}{\partial \dot{q}} = -p. \tag{60}$$

4 Classical Perturbation Theory

4.1 Regular and singular perturbation series

From regular perturbation problem, the approximation solution have a finite radius of convergence and the solution tends to the zeroth order results in the limit of $\epsilon \to 0$, onhte order of a power of ϵ (ϵ^2, ϵ^3, ... etc.) But in the singular perturbation problem, the expansion may not take the form of a power series, but if it does it will have a vanishing radius of convergence. A simple algebraic example is

$$\epsilon x^2 + x - 1 = 0. \tag{61}$$

Obviously, there are two solutions to the quadratic equation:

$$x^* = \frac{1}{2\epsilon}\left(-1 \pm \sqrt{1+4\epsilon}\right). \tag{62}$$

When $\epsilon \to 0$, one of the x^* is not well defined. But further investigation leads to

$$x = \frac{1}{2\epsilon}\left(-1 \pm 1 + 2\epsilon - 2\epsilon^2 + ...\right) \tag{63}$$

So there is a regular solution

$$x_1^* = 1 - \epsilon + ... \tag{64}$$

and there is a sigularity associated with the second root, which goes to ∞ in the limit of $\epsilon \to 0$:

$$x_2^* \sim -\frac{1}{\epsilon} - 1 + \epsilon + ... \tag{65}$$

References

[1] Arnol'd, V.I. (1992) *Oedinary Differential Equations.* (Cooke, R. Translate) Springer-Verlag, New York.

[2] Baker G.L. and Gollub, J.P. (1996) *Chaotic Dynamics: An Introduction.* Cambridge University Press, New York.

[3] Hirsch, M.W., Smale, S. and Devaney, R.L. (2004) *Differential Equations, Dynamical Systems & An Introduction to Chaos.* Elsevier Academic Press, New York.

[4] Ozorio de Almeida, A.M. (1988) *Hamiltonian Systems: Chaos and Quantization.* Cambridge Univ. Press, New York.

[5] Palis, J. and de Melo, W. (1982) *Geometric Theory of Dynamical Systems: An Introduction.* Springer-Verlag, New York.

[6] Perko, L. (2001) *Differential Equations and Dynamical Systems.* (3rd Ed.) Springer, New York.

[7] Strogatz, S.H. (2001) *Nonlinear Dynamics and Chaos: ith Applications to Physics, Biology, Chemistry, and Engineering.* Westview Press/Perseus Books Pub., New York.

[8] Tabor, M. (1989) *Chaos and Integrability in Nonlinear Dynamics: An Introduction.* Wiley-Interscience, New York.