AMATH 402
Methods for ordinary differential equations
Winter 2009
Homework 1: solutions

Due: Wednesday, January 14, 2009

1. Problem 2.1 (all parts) For 2.1.4, use the trig identity $\tan(x/2) = \sin x/(1 + \cos x)$.

2.1.1 The fixed points solve $\sin x = 0$, thus they are given by $X_k = k\pi$, for $k \in \mathbb{Z}$.

2.1.2 The velocity is $x' = \sin x$, thus the velocity is maximal when $\sin x = 1$, which occurs whenever x is one of $\{\ldots, -7\pi/2, -3\pi/2, \pi/2, 5\pi/2, 9\pi/2, \ldots\}$.

2.1.3 (a) The acceleration is $x'' = (\sin x)' = \cos x x' = \cos x \sin x = \sin(2x)/2$. (b) The acceleration is maximal when $\sin(2x) = 1$, which occurs when $2x$ is one of $\{\ldots, -7\pi/2, -3\pi/2, \pi/2, 5\pi/2, 9\pi/2, \ldots\}$, or when x is one of $\{\ldots, -7\pi/4, -3\pi/4, \pi/4, 5\pi/4, 9\pi/4, \ldots\}$.

2.1.4 Sidenote: let's prove the given trig identity:

$$
\begin{aligned}
\frac{\sin x}{1+\cos x} &= \frac{2\sin(x/2)\cos(x/2)}{1 + (2\cos^2(x/2) - 1)} \\
&= \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} \\
&= \frac{\sin(x/2)}{\cos(x/2)} \\
&= \tan(x/2).
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
& & t &= \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right| \\
&\Rightarrow & e^t &= \left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right| \\
&\Rightarrow & \pm e^t &= \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \\
&\Rightarrow & \frac{\pm e^t}{\csc x_0 + \cot x_0} &= \frac{1}{\csc x + \cot x}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & \frac{\pm e^t}{\frac{1}{\sin x_0} + \frac{\cos x_0}{\sin x_0}} = \frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} \\
\Rightarrow \qquad & \frac{\pm e^t \sin x_0}{1 + \cos x_0} = \frac{\sin x}{1 + \cos x} \\
\Rightarrow \qquad & \pm e^t \tan \frac{x_0}{2} = \tan \frac{x}{2}
\end{aligned}
$$

At this point, using $x = x_0$ at $t = 0$ gives $\pm \tan(x_0/2) = \tan(x_0/2)$, from which it follows we need to use the $+$ sign. Thus

$$\tan \frac{x}{2} = e^t \tan \frac{x_0}{2}.$$

(a) Using $x_0 = \pi/4$, we get

$$\tan \frac{x}{2} = e^t \tan \frac{\pi}{8}.$$

To calculate $\tan(\pi/8)$, we use the trig identity again: $\tan(\pi/8) = \sin(\pi/4)/(1 + \cos(\pi/4)) = \frac{\sqrt{2}/2}{1+\sqrt{2}/2} = \frac{\sqrt{2}}{2+\sqrt{2}} = 1/(\sqrt{2}+1)$. Thus

$$\tan \frac{x}{2} = \frac{e^t}{1+\sqrt{2}}.$$

Given the initial condition, we know x is always confined between the two fixed points $X_0 = 0$ and $X_1 = \pi$. Thus

$$\frac{x}{2} = \arctan \frac{e^t}{1+\sqrt{2}} \Rightarrow x = 2 \arctan \frac{e^t}{1+\sqrt{2}}.$$

(b) For $x_0 \in (X_k = k\pi, X_{k+1} = (k+1)\pi)$, we need to take into account the range of the $\arctan$ function: we get

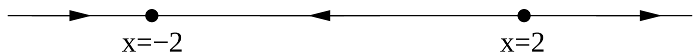
$$x = 2k\pi + 2 \arctan \left(e^t \tan \frac{x_0}{2} \right)$$

2. Problems 2.2.1, 2.2.3, 2.2.4, 2.2.7.

2.2.1 The fixed points satisfy $4x^2 - 16 = 0$, from which $x_1 = -2, x_2 = 2$. These are the only two fixed points. The analytical solution may be obtained using separation of variables:

$$
\begin{aligned}
& \int \frac{1}{4x^2 - 16} dx = t + c \\
\Rightarrow \qquad & -\frac{1}{8} \text{arctanh} \frac{x}{2} = t + c \\
\Rightarrow \qquad & \text{arctanh} \frac{x}{2} = -8(t + c) \\
\Rightarrow \qquad & \frac{x}{2} = -\tanh(8(t + c)) \\
\Rightarrow \qquad & x = -2 \tanh(8(t + c)).
\end{aligned}
$$

The phase portrait is shown below:



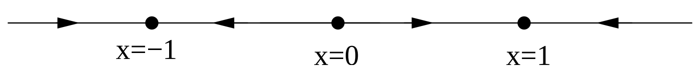
This shows that $x_1 = -2$ is stable, whereas $x_2 = 2$ is unstable.

2.2.3 The fixed points satisfy $x - x^3 = 0$, from which $x_1 = -1, x_2 = 0, x_3 = 1$. These are the only three fixed points. The analytical solution may be obtained using the following trick:

$$
\begin{aligned}
& & \frac{x'}{x^3} &= \frac{1}{x^2} - 1 \\
&\Rightarrow & -\frac{1}{2}\left(\frac{1}{x^2}\right)' &= \frac{1}{x^2} - 1 \quad (\text{let } z = 1/x^2) \\
&\Rightarrow & -\frac{1}{2}z' &= z - 1 \\
&\Rightarrow & z' + 2z &= 2 \\
&\Rightarrow & (ze^{2t})' &= 2e^{2t} \\
&\Rightarrow & ze^{2t} &= e^{2t} + c \\
&\Rightarrow & \frac{1}{x^2} &= 1 + ce^{-2t},
\end{aligned}
$$

which is not terribly instructive. We could get a more explicit solution using square roots.

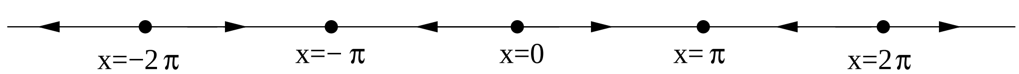
The phase portrait is shown below:



This shows that $x_1 = -1$ and $x_3 = 1$ are stable, whereas $x_2 = 0$ is unstable. A plot of the solutions is shown below (x on the vertical axis, t on the horizontal axis).

2.2.4 The fixed points satisfy $e^{-x}\sin(x)$, from which $x_k = k\pi$, with $k \in \mathbb{Z}$ are all fixed points. Thus there's an infinite number of fixed points. The analytical solution may be obtained using separation of variables, but results in an integral I can't do. Oh well.

The phase portrait is shown below:



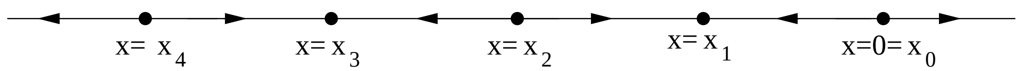
This shows that x_k with k odd is stable, whereas x_k with k even is unstable. A plot of the solutions is shown below (x on the vertical axis, t on the horizontal axis).

Note that for x negative, the solutions approach the stable equilibrium solutions very fast, whereas this approach is slow for x positive. This is all due to the exponental in the system.

2.2.7 The fixed points satisfy $e^x = \cos(x)$, which cannot be solved analytically. To get an idea of where the solutions might be, we plot e^x and $\cos(x)$ separately:

There's a solution at $x = 0$, and an infinite number of negative solutions. Let's label these $x_1 = 0$, $x_2 =$ the first negative solution, and so on. The analytical solution may be obtained using separation of variables, but again results in an integral I can't do. Oh well.

The phase portrait is shown below:



This shows that x_k with k odd is unstable, whereas x_k with k even is stable. A plot of the solutions is shown below (x on the vertical axis, t on the horizontal axis).

3. Problem 2.2.9. There are two equilibrium solutions: one at $x = 1$ (unstable) and another one at $x = 0$ (stable). We could use

$$x' = ax(x-1),$$

which has the right equilibrium points. In order to get the stability properties right, we need $a > 0$.

4. Problem 2.2.10.

   a) $x' = 0$
   b) $x' = \sin(\pi x)$
   c) It is not possible to pick three distinct points on a line and have all arrows nearby pointing to them.
   d) $x' = 1$
   e) $x' = (x-1)(x-2)\ldots(x-99)(x-100)$

5. Problem 2.2.13, parts a,b,c.

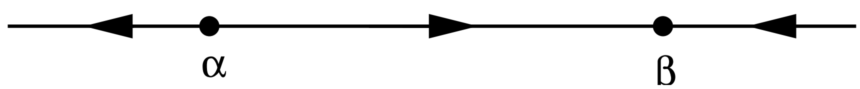
a) Using separation of variables:

$$
\begin{aligned}
&& mv' &= mg - kv^2 \\
\Rightarrow && \int_0^v \frac{mdv}{mg - kv^2} &= \int_0^t dt \\
\Rightarrow && \frac{m}{k}\int_0^v \frac{dv}{\frac{mg}{k} - v^2} &= t \\
\Rightarrow && \frac{m}{k}\sqrt{\frac{k}{mg}}\text{arctanh}\frac{v}{\sqrt{\frac{mg}{k}}} &= t \\
\Rightarrow && \text{arctanh}\frac{v}{\sqrt{\frac{mg}{k}}} &= \frac{k}{m}\sqrt{\frac{mg}{k}}t \\
\Rightarrow && \frac{v}{\sqrt{\frac{mg}{k}}} &= \tanh\left(\frac{k}{m}\sqrt{\frac{mg}{k}}t\right) \\
\Rightarrow && v &= \sqrt{\frac{mg}{k}}\tanh\left(\frac{k}{m}\sqrt{\frac{mg}{k}}t\right).
\end{aligned}
$$

b)

$$\lim_{t\to\infty} v = \sqrt{\frac{mg}{k}}.$$

c) The two fixed points are $v_1 = \alpha = -\sqrt{mg/k}$ and $v_2 = \beta = \sqrt{mg/k}$. The phase portrait is shown below.



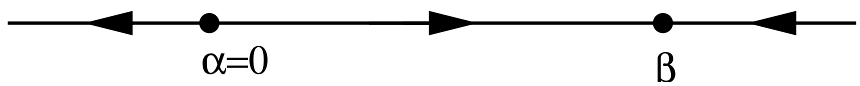
We conclude that $v_2 = \beta$ is stable, and thus the terminal velocity is $\sqrt{mg/k}$. You should also note that what happens to the left of $v = \alpha$ is an indication of the limitations of the model.

6. Problem 2.3.1, part b.

$$
\begin{aligned}
& & N' &= rN\left(1-\frac{N}{K}\right)\\
\Rightarrow & & \frac{N'}{N^2} &= r\left(\frac{1}{N}-\frac{1}{K}\right)\\
\Rightarrow & & -\left(\frac{1}{N}\right)' &= r\left(\frac{1}{N}-\frac{1}{K}\right)\\
\Rightarrow & & -x' &= r(x-1/K)\\
\Rightarrow & & x'+rx &= r/K\\
\Rightarrow & & \left(e^{rt}x\right)' &= \frac{r}{K}e^{rt}\\
\Rightarrow & & e^{rt}x &= \frac{1}{K}e^{rt}+c\\
\Rightarrow & & x &= \frac{1}{K}+ce^{-rt}\\
\Rightarrow & & \frac{1}{N} &= \frac{1}{K}+ce^{-rt}\\
\Rightarrow & & \frac{1}{N} &= \frac{1+cKe^{-rt}}{K}\\
\Rightarrow & & N &= \frac{K}{1+cKe^{-rt}}.
\end{aligned}
$$

7. Problem 2.3.2.

(a) The fixed points satisfy $k_1ax - k_{-1}x^2 = 0$, so that $x = \alpha = 0$ and $x = \beta = k_1a/k_{-1}$ are the two fixed points. The phase portrait is shown below.



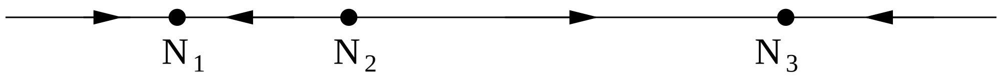
It follows from this that the fixed point $x_2 = \beta = k_1a/k_{-1}$ is stable, whereas the other one is unstable. Note that the figure should be ignored to the left of $x = 0$.

(b) Shown below for $k_1 = k_{-1} = a = 1$ (x on the vertical axis, t on the horizontal axis):

8. Problem 2.3.4.

a) The growth rate is $N'/N = r - a(N-b)^2$. In order for this to be maximal, its its derivative needs to be zero, and its second derivative needs to be negative. Thus $(N'/N)' = -2a(N-b)$ is zero at $N = b$. Its second derivative is $(N'/N)'' = -2a$. Thus, in order to have the Allee effect we need $b > 0$ and $a > 0$. Further, in order to have positive fixed points we also need $r > 0$.

b) The fixed points satisfy $N = 0$ or $r = a(N-b)^2$, which gives $N = 0$ or $N = b \pm \sqrt{r/a}$. There's two different possibilities: (i) if $b > \sqrt{r/a}$, there are three fixed points: $N_1 = 0$, $N_2 = b - \sqrt{r/a}$, and $N_3 = b + \sqrt{r/a}$; (ii) if $b \leq \sqrt{r/a}$ there are two fixed points: $N+_1 = 0$ and $N_2 = b + \sqrt{r/a}$. Note that because of the application we do not care about the negative fixed point. This second scenario gives rise to a model that behaves like the logistic equation and has no Allee effect. In what follows we proceed with the first scenario of three positive or zero fixed points. The phase portrait is shown below.



We see that the fixed point $N_1 = 0$ is stable, whereas N_2 is unstable. N_3 is stable.

c)

d) The qualitative difference is that now a minimal number of organisms need to be present to ensure growth to the carrying capacity. If the population is smaller than this critical number, the population dies out.