AMATH 402
Introduction to Dynamical Systems and Chaos
Winter 2009
Homework 3: Solutions

Due: Wednesday, January 28, 2009

1. 4.1.1

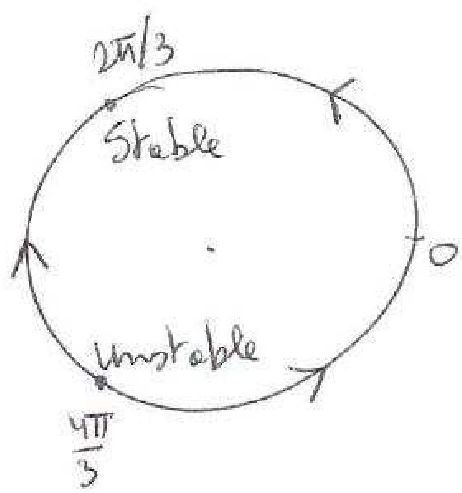
   $\sin(a\theta)$ has to be 2π periodic. This means that $a \in \mathbb{Z}$, i.e., an integer.

2. 4.1.2 Equation

$$\theta' = 1 + 2\cos\theta$$

   has fixed points at

$$\cos\theta^* = -\frac{1}{2} \quad \Rightarrow \quad \theta_1^* = \frac{2\pi}{3} \text{ stable}, \quad \theta_2^* = \frac{4\pi}{3} \text{ unstable}.$$



3. 4.1.4 Equation

$$\theta' = \sin^3\theta$$

   has fixed points at

$$\sin^3\theta^* = 0.$$

   That is $\theta_1^* = 0$ which is unstable, and $\theta_2^* = \pi$ which is stable.

4. 4.1.6 Equation

$$\theta' = 3 + \cos(2\theta)$$

has its right-hand-side

$$3 + \cos(2\theta) > 0 \quad \forall \theta \in [0, 2\pi).$$

Therefore, there is no fixed point on the cycle. The motion according to the equation is a continuous rotation.

5. 4.3.2

(a)

$$\begin{aligned} u &= \tan\left(\frac{\theta}{2}\right), \\ \theta &= 2\arctan u, \\ d\theta &= \frac{2du}{1+u^2}. \end{aligned}$$

One can also obtain the last equation by

$$\begin{aligned} du &= d\left\{\tan\left(\frac{\theta}{2}\right)\right\} = \sec^2\left(\frac{\theta}{2}\right)\frac{d\theta}{2}, \\ d\theta &= 2\sec^{-2}\left(\frac{\theta}{2}\right) du = \frac{2du}{(1+\tan^2(\theta/2))} = \frac{2du}{(1+u^2)} \end{aligned}$$

(b)

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} = \frac{2u}{1+u^2}.$$

(c)

$$\theta = \pm\pi \Rightarrow \frac{\theta}{2} = \pm\frac{\pi}{2} \Rightarrow u = \tan\frac{\theta}{2} = \tan\left(\pm\frac{\pi}{2}\right) = \pm\infty.$$

Therefore, the integration

$$\int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta}$$

can be written as

$$\int_{-\infty}^{\infty} \frac{\frac{2du}{1+u^2}}{\omega - a\frac{2u}{1+u^2}} = \int_{-\infty}^{\infty} \frac{2du}{\omega(1+u^2) - 2au}.$$

(d),(e) Therefore, the T on page 98

$$\begin{aligned} T &= \int_{-\infty}^{\infty} \frac{2du}{\omega(1+u^2)-2au} \\ &= \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{2du}{1+u^2-2au/\omega} \\ &= \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{2du}{u^2-2au/\omega+(a/\omega)^2+1-(a/\omega)^2} \\ &= \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{d(u-a/\omega)}{(u-a/\omega)^2+(1-(a/\omega)^2)} \end{aligned}$$

According to the result in 4.3.1, if $a/\omega < 1$, we thus have

$$T = \frac{2\pi}{\omega\sqrt{1-(a/\omega)^2}} = \frac{2\pi}{\sqrt{\omega^2-a^2}}.$$

6. 4.3.3

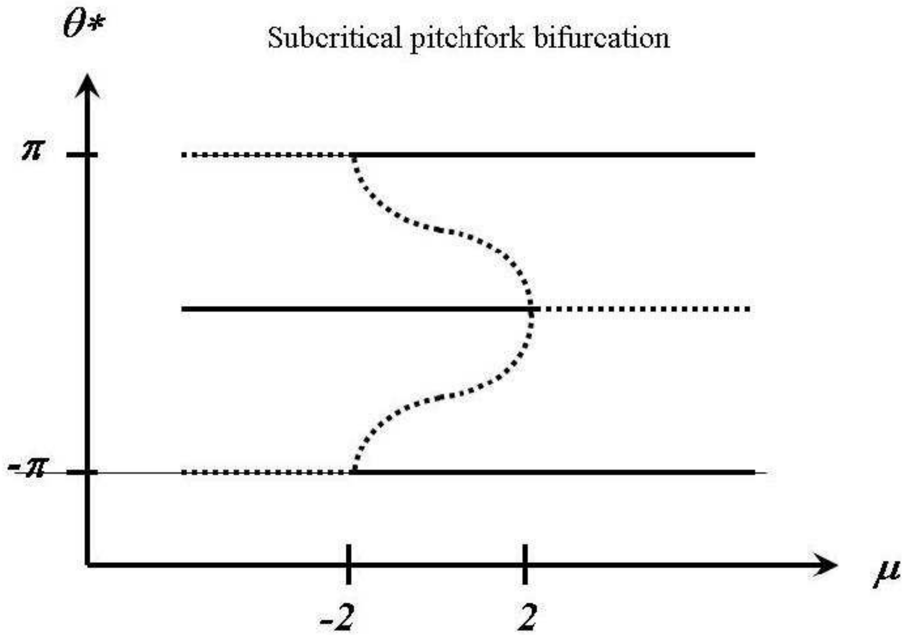
Equation

$$\begin{aligned} \dot{\theta} &= \mu\sin\theta - \sin(2\theta) \\ &= \mu\sin\theta - 2\sin\theta\cos\theta \\ &= \sin\theta\left(\mu - 2\cos\theta\right). \end{aligned}$$

The fixed points are at

$$\begin{aligned} &\sin\theta = 0, \Rightarrow \theta = 0, \pi; \\ &\cos\theta = \frac{\mu}{2}, \Rightarrow \theta = \pm\arccos\left(\frac{\mu}{2}\right) \text{ if } |\mu| < 2 \end{aligned}$$

At $\mu = \pm 2$ there are bifurcation values.



7. <u>4.3.4</u> Equation

$$\theta' = \frac{\sin\theta}{\mu + cos\theta}$$

has both fixed point(s) where $\theta' = 0$ and singularity where $\theta' = \infty$.

For $|\mu| > \mathbf{1}$, the denominator $\mu + \cos\theta$ does not change sign. Hence, the fixed points are at $\sin\theta = 0$. They are $\theta_1^* = 0$ and $\theta_2^* = \pi$.

$$\theta_1^* = 0 \text{ stable, and } \theta_2^* = \pi \text{ unstable if } \mu < -1,$$

$$\theta_1^* = 0 \text{ unstable, and } \theta_2^* = \pi \text{ stable if } \mu > 1.$$

For $|\mu| < 1$, there are two roots for the denominator $\mu + \cos\theta = 0, \Rightarrow \theta_1^\ddagger$ and $\theta_2^\ddagger$. We shall denote $\theta_1^\ddagger \in (0, \pi)$ and $\theta_2^\ddagger \in (\pi, 2\pi)$.

We now analyze the singularity $\theta_1^\ddagger$. For $\theta < \theta_1^\ddagger$, $\frac{\sin\theta}{\mu+\cos\theta} > 0$ because the numerator is positive and $\cos\theta$ is a decreasing function. But for $\theta > \theta_1^\ddagger$, $\frac{\sin\theta}{\mu+\cos\theta} < 0$. Therefore, the singularity $\theta_1^\ddagger$ is an attractor. Near the singularity $\theta_2^\ddagger$, $\frac{\sin\theta}{\mu+\cos\theta} > 0$ for $\theta < \theta_2^\ddagger$ and $\frac{\sin\theta}{\mu+\cos\theta} > 0$ for $\theta < \theta_2^\ddagger$, so again the singularity $\theta_2^\ddagger$ is also an attractor (see figure in which the black squares reprent the singularities.)

For $\mu = -1$, the fixed point $\theta_2^* = \pi$ is unstable. The fixed point $\theta_1^* = 0$ and singularities $\theta_1^\ddagger$, $\theta_2^\ddagger$ merge:

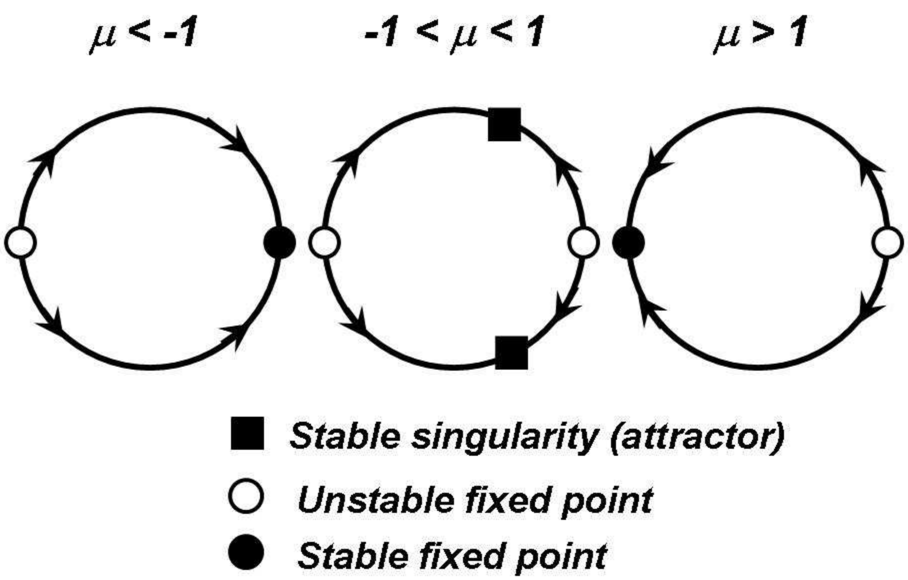
$$\frac{\sin\theta}{-1 + \cos\theta} \approx \frac{\theta - \theta^3/3!}{-\theta^2/2 + \ldots} \approx -\frac{2}{\theta},$$

which is an attractive singular point.

For $\mu = +1$, the fixed point $\theta_1^* = 0$ is unstable. fixed point $\theta_2^* = \pi$ and singularity $\theta_1^\ddagger$, $\theta_2^\ddagger$ merge. Let $\phi = \theta - \pi$, then

$$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin(\phi+\pi)}{1+\cos(\phi+\pi)} = \frac{-\sin(\phi)}{1-\cos(\phi)} \approx \frac{-\phi+\phi^3/3!}{\phi^2/2+\ldots} \approx -\frac{2}{\phi},$$

which is again an attractive singular point.



8. <u>4.3.5</u> Equation

$$\theta' = \mu + \cos\theta + \cos(2\theta)$$

can be re-written as

$$\theta' = (\mu - 1) + \cos\theta + 2\cos^2\theta. \tag{1}$$

If $\mu > 9/8$, i.e., $\mu - 1 > 1/8$, then $f(x) = \mu - 1 + x + 2x^2 > 0$ for all x. Hence the right-hand-side of Eq. (1) is positive for all θ. There is no fixed point; the motion is a continuous counter-clockwise rotation.

If $\mu < -2$, i.e., $\mu - 1 < -3$, then the two roots of $f(x)$, x_1^* and $x_2^* \notin \left[-\frac{3}{2}, 1\right] \Rightarrow$ no fixed point for Eq. (1). The motion is a continuous clockwise rotation.

For $-2 < \mu < 0$, the larger one of the two roots of $f(x)$, $x_2^* \in \left(\frac{1}{2}, 1\right)$, while the $x_1^* < -1$:

$$x_2^* = \frac{-1+\sqrt{1-8(\mu-1)}}{4}.$$

Therefore, Eq. (1) has two fixed points

$$\cos\theta = x_2^*.$$

For $0 < \mu < 9/8$, both roots of $f(x)$, x_1^*, x_2^* are now $\in (-1, 1)$: $-1 < x_1^* < -\frac{1}{4} < x_2^* < 1$. This corresponds to four fixed points.

At $\mu = -2$, there is a saddle-node bifurcation at $\theta = 0\ (2\pi)$. **At** $\mu = 0$, there is a saddle-node bifurcation at $\theta = \pi$. **At** $\mu = 5/8$, there are two simultaneous saddle-node bifurcations, at $\arccos\left(-\frac{1}{4}\right)$, they are at $\theta = 104.5^o$ and $\theta = 225.5^o$.