**AMath 403 Spring 2004**
**Final exam review**

# 1 Course review

1. Separation of variables; Fourier series, Fourier sine series, Fourier cosine series; eigenfunction expansion

   (a) These methods are used on a *finite domain.*

   (b) The solution is separated into functions of single variables, e.g., $u(x,t) = X(x)T(t)$. The PDE is then written as multiple ODEs. (This is not always possible!) The ODE with homogeneous boundary conditions is called the *eigenvalue problem.* (This is usually in $x$.) The general solution is written then as a sum of eigenfunctions with coefficients chosen to satisfy the non-homogeneous boundary or initial conditions.

   (c) For homogeneous Dirichlet boundary conditions, i.e., $u(0,t) = u(L,t) = 0$, the eigenfunction are sines. This gives *Fourier sine series.* The Fourier sine series for $f(x)$ on $0 < x < L$ is

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

   (d) For homogeneous Neumann boundary conditions, i.e., $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$, the eigenfunction are cosines. This gives *Fourier cosine series.* The Fourier cosine series for $f(x)$ on $0 < x < L$ is

$$f(x) = \sum_{n=0}^{+\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$
$$a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x,$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \quad \text{for} \quad n = 1, 2, \ldots.$$

(e) For periodic boundary conditions, i.e., $u(-L,t) = u(L,t)$ and $\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$, the eigenfunction are sines and cosines. This gives *Fourier series*. The Fourier series for $f(x)$ on $-L < x < L$ is

$$\begin{aligned}
f(x) &= \sum_{n=0}^{+\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right), \\
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,\mathrm{d}x, \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad \text{for} \quad n = 1, 2, \ldots, \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.
\end{aligned}$$

(f) For *non-homogeneous* PDEs, use eigenfunction expansion. That is, write every function $f(x,t)$ in a series of the eigenfunctions for the *homogeneous* problem, $\phi_n$, i.e., $f(x,t) = \sum \alpha_n(t)\phi_n(x)$. This gives an ODE for $\alpha_n(t)$

(g) All of these methods can only deal with *one non-homogeneity* at a time! To solve a problem with multiple non-homogeneity, solve the problems with only *one non-homogeneity* at a time and then add the solutions.

2. Higher-dimensional PDE; Sturm–Liouville problems

(a) Use the same idea as for two-dimensional problems but several times, e.g., $u(x,y,t) = U(x,y)T(t)$ and then $U(x,y) = X(x)Y(y)$. This gives $n$ ODEs for an $n$-dimensional PDE.

(b) For *non-rectangular* coordinates the eigenfunctions are not sines and cosines. Sturm–Liouville theory says solutions to the PDE can be written as sums of the eigenfunctions.

(c) *Polar coordinates* and *cylindrical coordinates* give Bessel's equation,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left(\lambda r - \frac{m^2}{r}\right)R = 0.$$

The solutions are Bessel functions, $J_m\left(\sqrt{\lambda}r\right)$ and $Y_m\left(\sqrt{\lambda}r\right)$. ($Y_m$ is unbounded at $r = 0$ so only $J_m$ is used on problems where $r = 0$ is inside the domain.)

(d) *Spherical coordinates* give the spherical Bessel's equation,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}R}{\mathrm{d}\rho}\right) + \left[\lambda\rho^2 - m(m+1)\right]R = 0.$$

The solutions are spherical Bessel functions, $j_m\left(\sqrt{\lambda}r\right)$ and $n_m\left(\sqrt{\lambda}r\right)$. ($n_m$ is unbounded at $\rho = 0$ so only $j_m$ is used on problems where $r = 0$ is inside the domain; we didn't talk about $n_m$ in class, but it analogous to $Y_m$ for polar coordinates.)

3. Fourier transforms

(a) This method is used on an *infinite domain*; it is the analog of Fourier series.

(b) The Fourier transform and inverse Fourier transform are defined by

$$\mathcal{F}[f] = \hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}x,$$

$$\mathcal{F}^{-1}\left[\hat{f}\right] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}\omega.$$

(c) The Fourier transform of derivatives gives

$$\mathcal{F}\left[\frac{\partial^n u}{\partial x^n}\right] = (-\mathrm{i}\omega)^n\hat{u},$$

$$\mathcal{F}\left[\frac{\partial^n u}{\partial t^n}\right] = \frac{\partial^n \hat{u}}{\partial t^n}.$$

(d) The Fourier transform of a PDE gives an ODE. Solving the ODE and then computing the inverse Fourier transform gives the solution.

4. Green's functions

(a) Green's functions can be used on either finite or infinite domains. Green's functions give the solution to a PDE in terms of the integral of non-homogeneities with the Green's function.

(b) To problem to solve to find the Green's function is the *adjoint problem*, with both the *adjoint operator* and the *adjoint* boundary and initial conditions (which are found from the *homogeneous* boundary and initial conditions on the original problem).

(c) On infinite domains, Green's functions can be found by radial symmetry or Fourier transforms. These are called *free-space Green's functions*. Often we just look these up in books.

(d) The *method of images* finds Green's functions for *semi-infinite* or *finite* domains from free-space Green's functions by geometric arguments.

(e) Eigenfunction expansions can be used to find Green's functions on *finite* domains.

(f) Boundary and initial conditions can be incorporated by looking at the remaining terms from computing the adjoint.

5. Characteristics

(a) The method of characteristics is used for solving *first-order quasi-linear* PDE, i.e.

$$\frac{\partial u}{\partial t} + c(u, x, t)\frac{\partial u}{\partial x} = f(u, x, t).$$

(b) The method is to look for curves $(x(t), t)$ on which the PDE becomes an ODE. This results in the system of ODE

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= f(u, x, t), \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= c(u, x, t), \end{aligned}$$

with the initial conditions $u(x_0, 0) = u_0(x_0)$ and $x(0) = x_0$.

(c) The solution for $x$, $x = X(t, x_0)$, gives the *characteristic curves*. If the curves do not intersect, the solution is valid for all time. The point at which the curves intersect can be found by look for the first point at which $\frac{\partial u}{\partial x}$ or $\frac{\partial u}{\partial t}$ is infinite.

(d) If the characteristic curves intersect, a *shock solution* can be constructed using an *integral conservation law.*

(e) If there are no characteristics in a region, a solution can be constructed by inserting a *fan* into the region.

# 2 Exam review problems

1. Solve the Schrödinger equation in 1 spatial dimension,

$$\mathrm{i}\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

with

$$u(0,t) = u(L,t) = 0,$$

and an arbitrary initial condition.

2. Solve the Schrödinger equation in 2 spatial dimensions,

$$\mathrm{i}\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right], \quad 0 < r < a, \quad t > 0,$$

with

$$u(a,\theta,t) = 0,$$

and an arbitrary initial condition.

3. Solve

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, \quad t > 0,$$

with

$$u(x,0) = \delta(x).$$

4. Find the Green's function solution for

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < +\infty, \quad t > 0,$$

with

$$u(0,t) = g(t),$$

and

$$u(x,0) = h(x),$$

given that the free-space Green's function is

$$G_{FS}(x,t;x_0,t_0) = \frac{H(t_0 - t)}{\sqrt{4\pi(t_0 - t)}}\, \mathrm{e}^{-\frac{(x-x_0)^2}{4(t_0-t)}}.$$

First, find the Green's function for the semi-infinite domain by the method of images and then figure out what the extra terms are needed for the non-homogeneous boundary and initial conditions.

5. Solve

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = \frac{1}{u}, \quad -\infty < x < +\infty, \quad t > 0,$$

with

$$u(x,0) = 0.$$