AMath 403 Spring 2004
Homework Assignment 6
Assigned: Friday, 21.May, 2004
Due in class: Friday, 28.May, 2004

1. In this problem we will find the Green's function for *Poisson's equation* in three dimensions:

$$Lu = \nabla^2 u = f(x, y, z), \quad -\infty < x, y, z < +\infty.$$

   (a) What is the adjoint operator, L^*? What is the PDE for the Green's function, $G(x, y, z; x_0, y_0, z_0)$?

   (b) Let $\vec{x} = (x, y, z)$, $\vec{x}_0 = (x, y, z)$, $\vec{\rho} = \vec{x} - \vec{x}_0$, and $g(\vec{\rho}) = G(\vec{x}; \vec{x}_0)$. Now what is the PDE for the Green's function, $g(\vec{\rho})$?

   (c) Argue that the PDE can be simplified to a second-order ODE in ρ, the radius in spherical coordinates, for the Green's function. Solve the ODE for $\rho > 0$.

   (d) Integrate the ODE inside of the ball of radius ϵ and use the limit of this integral as $\epsilon \to 0$ to find one constant of integration. Use the normalization condition $g(\rho) \to 0$ as $\rho \to +\infty$ to find the second constant.

   (e) Write the Green's function in terms of the original variables x, y, z, x_0, y_0, and z_0. What is the solution to the original problem for $u(x, y, z)$?

2. In this problem we will find the Green's function for the *heat equation* on a finite domain by using eigenfunction expansions:

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad 0 < x < 1, \quad t > 0,$$

with

$$\frac{\partial u}{\partial x}(0, t) = g(t), \quad \frac{\partial u}{\partial x}(1, t) = h(t), \quad \text{and} \quad u(x, 0) = j(x).$$

   (a) What is the adjoint operator, L^*? What are the *homogeneous* boundary and 'initial' conditions for the adjoint problem? What are the PDE, boundary conditions, and 'initial' condition for the Green's function, $G(x, t; x_0, t_0)$?

(b) What are the eigenfunctions for the PDE? (Remember, the eiegnfunctions are in x.) Write $G(x,t;x_0,t_0)$ in an eigenfunction expansion, $G(x,t;x_0,t_0) = \sum_n A_n(t;x_0,t_0)\phi_n(x)$.

(c) Plug the expansion for G into the PDE for the adjoint problem. What is the eigenfunction expansion of the right-hand side? What is the resulting ODE for $A_n(t;x_0,t_0)$?

(d) Solve the ODE for A_n in two pieces, one with $t > t_0$ and one with $t < t_0$ and apply the 'initial' condition. Integrate the ODE from $t = t_0 - \epsilon$ to $t = t_0 + \epsilon$ and use the limit as $\epsilon \to 0$ to find the jump condition on A_n. Use this to find the remaining constant for the solution for A_n.

(e) What is the solution for $G(x,t;x_0,t_0)$?

(f) Use integration by parts (in both x and t) to find the solution, $u(x,t)$, with the *non-homogeneous* boundary and initial conditions.