Matrices

Think of matrices as $N \times M$ tables of numbers N rows, M columns

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

entries A(n,m)

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>> A=[1,2,3 ; 4 6 7 ; 1 3 4]

A =
       1       2       3
       4       6       7
       1       3       4


>> A23=A(2,3)
A23 =      7
```

N-element col. vector: N, M=1

M-element row. vector: N=1, M

Otherwise, we will mostly consider square matrices (N=M)

FUNDAMENTAL CONCEPT:

Matrix-vector multiplication

Matrices perform operations on vectors: linear combinations of vector elements

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{1,1}x_1 + A_{1,2}x_2 \\ A_{2,1}x_1 + A_{2,2}x_2 \end{pmatrix}$$

e.g.

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

General formula:

$[A\vec{x}]_i = \sum_j A_{ij}x_j$

Another interpretation: note,

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} A_{1,1} \\ A_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} A_{1,2} \\ A_{2,2} \end{pmatrix}$$

In general,

$$\begin{pmatrix} | & \cdots & | \\ a_1 & \cdots & a_n \\ | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_j x_j \begin{pmatrix} | \\ a_j \\ | \end{pmatrix}$$

MATLAB * operator

```
>> A=[1 1 ; 1 2] ; A*[1 ; 2]

ans =

     3
     5
```

In $y = Ax$, A must have same number of rows.

Nonsense:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

```
>> A=[1 1 ; 1 2] ; A*[1 ; 2 ; 4]
??? Error using ==> mtimes
Inner matrix dimensions must agree.
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Inner dimensions:

A has $N_A\ (rows) \times M_A\ (columns)$

$\vec{x}$ has $N_x\ (rows) \times 1\ (columns)$

Write: $A_{N_A \times M_A} x_{N_x \times 1}$

Need: $M_A = N_x$, i.e., length of vectors is number of columns in matrix

MATRIX - MATRIX multiplication.

Do two multiplications in order: $\vec{y} = A(B\vec{x})$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \left(\begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} B_{1,1}x_1 + B_{1,2}x_2 \\ B_{2,1}x_1 + B_{2,2}x_2 \end{pmatrix} = \begin{pmatrix} A_{1,1}\left(B_{1,1}x_1 + B_{1,2}x_2\right) + A_{1,2}\left(B_{2,1}x_1 + B_{2,2}x_2\right) \\ A_{2,1}\left(B_{1,1}x_1 + B_{1,2}x_2\right) + A_{2,2}\left(B_{2,1}x_1 + B_{2,2}x_2\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(A_{1,1}B_{1,1} + A_{1,2}B_{2,1}\right)x_1 + \left(A_{1,1}B_{1,2} + A_{1,2}B_{2,2}\right)x_2 \\ \left(A_{2,1}B_{1,1} + A_{2,2}B_{2,1}\right)x_1 + \left(A_{2,1}B_{1,2} + A_{2,2}B_{2,2}x_1\right)x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(A_{1,1}B_{1,1} + A_{1,2}B_{2,1}\right) & \left(A_{1,1}B_{1,2} + A_{1,2}B_{2,2}\right) \\ \left(A_{2,1}B_{1,1} + A_{2,2}B_{2,1}\right) & \left(A_{2,1}B_{1,2} + A_{2,2}B_{2,2}\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Define matrix multiplication AB so that $A(B\vec{x}) = (AB)x$

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} = \begin{pmatrix} \left(A_{1,1}B_{1,1} + A_{1,2}B_{2,1}\right) & \left(A_{1,1}B_{1,2} + A_{1,2}B_{2,2}\right) \\ \left(A_{2,1}B_{1,1} + A_{2,2}B_{2,1}\right) & \left(A_{2,1}B_{1,2} + A_{2,2}B_{2,2}\right) \end{pmatrix}$$

In general: $[AB]_{i,j} = \sum_k A_{i,k}B_{k,j}$.

“Sum across ith row of first matrix, down jth col of second.

VISUALIZE:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$[AB]_{ij} = \sum_k A_{i,k} B_{k,j} = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \end{bmatrix}$$

First application of matrix-vector manipulations: define systems of linear equations

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 4 & 3 \\ 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 4x_1 + 3x_2 \\ 5x_1 + 9x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$