A note on eigenvectors an eigenvalues

Say you are doing the exercise in Section 8 of the Ellner and Guckenheimer lab manual, and you are asked to find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ .3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .7 & .9 \end{pmatrix}$$

According to the lab manual, you should get for the dominant eigenvalue, and dominant left and right eigenvectors v and w:

Results: $\lambda = 1.0419$
$w = (0.6303, 0.1815, 0.0697, 0.0334, 0.0193, 0.0111)^T$
$v = (1, 3.4729, 9.0457, 18.8487, 32.7295, 56.8328, 84.5886)^T$

However, MATLAB returns:

```
>> [V,D]=eig(A)

V =

   0.8711              0.8711              0.8852              0.8852              0.9158              0.9158              0.9506
  -0.3301 - 0.1753i  -0.3301 + 0.1753i  -0.0387 - 0.3626i  -0.0387 + 0.3626i   0.2552 - 0.2174i   0.2552 + 0.2174i   0.2737
   0.1198 + 0.1771i   0.1198 - 0.1771i  -0.1958 + 0.0423i  -0.1958 - 0.0423i   0.0260 - 0.1615i   0.0260 + 0.1615i   0.1051
  -0.0162 - 0.1520i  -0.0162 + 0.1520i   0.0431 + 0.1306i   0.0431 - 0.1306i  -0.0518 - 0.0853i  -0.0518 + 0.0853i   0.0504
  -0.0489 + 0.1217i  -0.0489 - 0.1217i   0.1032 - 0.0467i   0.1032 + 0.0467i  -0.0694 - 0.0230i  -0.0694 + 0.0230i   0.0290
   0.0860 - 0.0726i   0.0860 + 0.0726i  -0.0473 - 0.0805i  -0.0473 + 0.0805i  -0.0496 + 0.0201i  -0.0496 - 0.0201i   0.0167
  -0.0448 + 0.0238i  -0.0448 - 0.0238i  -0.0057 + 0.0534i  -0.0057 - 0.0534i   0.0476 + 0.0406i   0.0476 - 0.0406i   0.0825


D =

  -0.6176 + 0.3279i        0                  0                  0                  0                  0                  0
        0            -0.6176 - 0.3279i        0                  0                  0                  0                  0
        0                  0            -0.0773 + 0.7241i        0                  0                  0                  0
        0                  0                  0            -0.0773 - 0.7241i        0                  0                  0
        0                  0                  0                  0             0.6239 + 0.5314i        0                  0
        0                  0                  0                  0                  0             0.6239 - 0.5314i        0
        0                  0                  0                  0                  0                  0             1.0419
```

Here, the diagonal entries of D are the eigenvalues. The corresponding columns of D are the associated eigenvectors. In particular the dominant eigenvalue is the last one in the list, $\lambda = 1.0419$. That checks against the Ellner and Guckenheimer lab manual. However, according to MATLAB the corresponding right eigenvector is the last column of V, which is

```
>> w=V(:,7)

w =

    0.9506
    0.2737
    0.1051
    0.0504
    0.0290
    0.0167
    0.0825
```

That does NOT match the given answer. What is going on ?

We need to remember that eigenvectors are defined only up to an arbitrary scale factor.

The defining equation for x to be an eigenvector of A with eigenvalue λ is

$$Ax = \lambda x \ .$$

But note that if x is an eigenvector with eigenvalue λ, so is $c\,x$, where c is any constant (even a negative number, say). Indeed, it passes the same test:

$$\begin{aligned} A(cx) &= c\ Ax \\ &= c\,\lambda\,x \\ &= \lambda\,(cx) \end{aligned}$$

If we use $c = 0.6303/0.9506$, we see that our eigenvector agrees with the one in the lab manual. That is,

```
>> w*0.6303/0.9506

ans =

    0.6303
    0.1815
    0.0697
    0.0334
    0.0193
    0.0111
    0.0547
```

which DOES give us the answer in the lab manual.

The point is that even though we talk about "the" dominant eigenvector, etc., this is only meaningful up to a scale constant. But, it is still useful:

1. The ratios of the entries in the dominant eigenvector are unaffected by scaling, and represent relative proportions in the various population stages.

2. In our formula for eigenvalue sensitivities, the scale constant will cancel out in the numerator and denominator – so we always get the same answer.