1. Function associated with random variable $\mathbf{X}$

$$F_{\mathbf{X}}(x) = \Pr\{\mathbf{X} \le x\}$$

   is called cumulative distribution function. Are the following statements correct? Why?

   (a) $0 \le F(x) \le 1$

   (b) $F(x)$ is monotonic

   (c) $F(x)$ is a continuous function of x

2. (a) Plot the cumulative distribution function for random variable $\mathbf{X}$

$$F_{\mathbf{X}}(x) = \begin{cases} 0 & x \le 0, \\ x^3 & 0 < x < 1, \\ 1 & x \ge 1. \end{cases}$$

   (b) Determine the corresponding probability density function $f_{\mathbf{X}}(x)$ in the three regions (i) $x \le 0$, (ii) $0 < x < 1$, and (iii) $1 \le x$.

   (c) What is the mean of the distribution?

   (d) Evaluate $\Pr\{\frac{1}{4} \le \mathbf{X} \le \frac{3}{4}\}$.

3. Let $\mathbf{X}$ be a Poisson random variable with mean λ:

$$P_{\mathbf{X}}(n) = \frac{\lambda^n}{n!}e^{-\lambda}, \qquad (n \ge 0)$$

   Determine $\Pr\{\mathbf{X} = \text{odd}\}$. What is the

$$E\left[\frac{1}{1+\mathbf{X}}\right]?$$

4. Let $F(x)$ be the cumulative distribution function of a continuous random variable $\mathbf{X}$. Assume that $\mathbf{X}$ is positive and finite. Show that the expected value

$$E[\mathbf{X}] = \int_0^\infty [1 - F(x)]dx.$$

   Be sure to note where to use the conditions $\mathbf{X}$ being positive and finite.

5. Suppose that the random $\mathbf{X}$ is exponentially distributed. Show that $\mathbf{X}$ has the following property:

$$\Pr\{\mathbf{X} \ge t + \tau | \mathbf{X} \ge t\} = \Pr\{\mathbf{X} \ge \tau\}.$$

   This property of the exponential distribution is known as the *memoryless property.*

6. The ***characteristic function*** of a continuous random variable $\mathbf{X}$ is defined as

$$Q_{\mathbf{X}}(z) = \int_{-\infty}^\infty e^{zs} f_{\mathbf{X}}(s)ds.$$

   (a) Show that the characteristic function of $\mathbf{Y} = \mathbf{X} - \mu_{\mathbf{X}}$ is $Q_{\mathbf{Y}}(z) = e^{-\mu_{\mathbf{X}} z} Q_{\mathbf{X}}(z)$.

   (b) Show that $Q_{\mathbf{Y}}^{(k)}(0) = E\left[\mathbf{Y}^k\right] = E\left[(\mathbf{X} - \mu_X)^k\right]$. $Q_{\mathbf{Y}}^{(k)}(z)$ is the kth derivative of function $Q_{\mathbf{Y}}(z)$ with respect to z. $E\left[(\mathbf{X} - \mu_X)^k\right]$ is called the kth moment of $\mathbf{X}$ about its expected value.

   (c) Suppose that $\mathbf{X}$ is a binomial random variable. Find $Q_{\mathbf{Y}}(z)$; then find the first three moments of $\mathbf{X}$ about its expected value.

7. Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables having cumulative distribution functions $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$, respectively. Define $\mathbf{W} = \min\{\mathbf{X}, \mathbf{Y}\}$ to be the smaller of the two. Show that $F_{\mathbf{W}}(w) = 1 - [1 - F_{\mathbf{X}}(w)][1 - F_{\mathbf{Y}}(w)]$ for all w.