1. Calculate the generating function for the binomial distribution

$$P_k = \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k}$$

and for the Poisson distribution

$$P_k = \frac{1}{k!} \lambda^k e^{-\lambda}.$$

Try to show that if $p = \lambda/N$, then the generating function for the former approaches to the latter when N is very large. (You need to know that for large N, $(1+1/N)^N \approx e$.)

2. N bacteria are spread independently with uniform distribution on a microscope slide of area A. An arbitrary region having area a is selected for observation. Determine the probability of k bacteria within the region of area a. Show that as $N \longrightarrow \infty$ and $a \longrightarrow 0$ such that $(a/A)N \longrightarrow c$ $(0 < c < \infty)$, then $p(k)$ is a Poisson distribution with mean c.

3. Try to show that if $\mathbf{X}_k$, $(k = 1, 2, ..., n)$ are n identical, independent exponentially distributed random variables with expectation τ, then

$$\mathbf{X}^* = min(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n)$$

is also an exponentially distributed random variable. What is the expectation for $\mathbf{X}^*$?

4. Find the stationary distribution $\boldsymbol{\mu}$, $\boldsymbol{\mu} = \boldsymbol{\mu}\mathbf{S}$, of a four-state Markov mode for DNA base:

$$\mathbf{S} \quad = \quad \begin{array}{c|cccc} & \text{A} & \text{C} & \text{G} & \text{T} \\ \hline \text{A} & .32 & .18 & .23 & .27 \\ \text{C} & .37 & .23 & .05 & .35 \\ \text{G} & .30 & .21 & .25 & .24 \\ \text{T} & .23 & .19 & .25 & .33 \end{array}$$

Verify your result by numerical iteration $\mathbf{S}^n$ with $n \to \infty$. What does this say about the frequency of the different bases? Any biological meanings?

5. Let matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}, \qquad 0 < a, b < 1.$$

Verify that it is a Markov matrix. Find $\mathbf{P}^n$. What is the the limit of $n \to \infty$? What happens if $a = b = 1$? Explain your result.

6. Suppose P is an $N \times N$ stochastic (Markov) matrix (row sums equal one),

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ p_{N1} & p_{N2} & \cdots & p_{NN} \end{pmatrix}.$$

(a) Show that P^2 is a stochastic (Markov) matrix. Then show that P^n is a stochastic (Markov) matrix for all positive integers n.

(b) Suppose P is a doubly stochastic matrix (row and colume sums equal one). Show that P^n is a doubly stochastic matrix for all positive interger n.

7. [Optional] The deterministic, discrete time and discrete state mathematical models for two populations of species, N_1 and N_2, are often in the form

$$N_1(t+1) = f(N_1(t), N_2(t)), \quad N_2(t+1) = g(N_1(t), N_2(t)) \tag{1}$$

in which f and g are two nonlinear functions. For example, N_1 and N_2 are the healthy and infectious populations in a model for epidemiology. Let us assume that these two equations can be transformed into a single equation for the fraction of N_1

$$p_1(t+1) = h(p_1(t), p_2(t)) \tag{2}$$

in which $p_i = N_i/(N_1 + N_2)$, $i = 1, 2$. Noting that $p_2(t) = 1 - p_1(t)$, then Eq. 2 is a discrete state and discrete time dynamical system.

Can one interpret the p_i as the probability of an individual in population i? Given an initial (p_1, p_2) at $t = 0$, Eq. 2 yields all the (p_1, p_2) for $t > 0$. Is the model in Eq. 2 a Markov model? When it is, what is the transition probability? When it is not, why and try to give an example.