1. Messages arrive at a telegraph office as a Poisson process with mean rate of 3 message per hour.

   (a) What is the probability that no message arrive during the morning hours 8:00AM to noon?

   (b) What is the distribution of the time at which the first afternoon message arrives?

2. Consider a branching process X_n whose offspring distribution is $\xi = 0, 2$ with respective probabilities q and $p = 1 - q$. Assume $X_0 = 1$. What is the probability of extinction? Compare the result with a biased random walk process Y_n with transition probabilities $\Pr\{Y_{n+1} = j + 1 | Y_n = j\} = p$ and $\Pr\{Y_{n+1} = j - 1 | Y_n = j\} = q$. Assume that the random walk starts at $Y_0 = 1$ and is absobed either at 0 or $N = \infty$. What is the relation between the extinction probability for the branching process and the exit probability at 0 for the random walk? Compare the results for $X_0 = Y_0 = k$.

3. Lotka in 1931 found that for white males in the US in 1920 the probability generating function of the number of male offspring per male is approximately given by

$$\phi(s) = \frac{0.482 - 0.041s}{1 - 0.559s}.$$

   (a) What is the mean number of offspring for each male? What is the probability of having zero son for a male?

   (b) What is the probability of extinction of a surname (descended from a single male)?

4. A chemical solution contains N molecules of type A and M molecules of type B. An irreversible reaction occurs between type A and B molecules in which they bond to form a new compound AB. Suppose that in any small time interval of length h, any particular unbonded A molecule will react with any particular unbound B molecule with probability $\theta h + o(h)$, where θ is a reaction rate constant. Let $X(t)$ denote the number of unbonded A molecules at time t.

   (a) Model $X(t)$ as a continuous-time Markov process by specifying the transition probability rate (Q) matrix.

   (b) Assume that $N < M$, so that eventually all of the A molecules become bonded. Determine the mean time until this happens.

   (c) How can you generalize the problem to a reversible reaction in which AB can fall apart to form A and B. Assume the life time of AB is exponentially distributed with mean time λ.

5. Compute the matrix exponential e^Q, which is defined as

$$e^Q = \sum_{n=0}^{\infty} \frac{1}{n!} Q^n,$$

   for

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

Try to compare the e^Q with the matrix

$$P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}, \qquad 0 < a, b < 1.$$

The P matrix, which appeared in homework # 2, is associated with a 2-state, discrete-time Markov chain. Q, on the other hand, is associated with a 2-state, continuous-time Markov Chain. Try to relate the two matrices in the context of Markov processes.

6. Let a continuous, positive RV $\mathbf{T}$ be the waiting time for the arrival of a product via an enzymatic reaction with a single enzyme molecule:

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} ES \xrightarrow{k_2} E + P.$$

In general, we have the pdf for $\mathbf{T}$

$$f_T(t) = ar_1 e^{-r_1 t} + (1-a) r_2 e^{-r_2 t} \tag{1}$$

where r_1 and r_2 are two positive constants. Being a pdf, show that $f_T(t)$ satisfies

$$\int_0^\infty f_T(t)dt = 1.$$

(a) Compute the conditional probability:

$$\Pr\{\mathbf{T} > t + \tau | \mathbf{T} > t\}.$$

(b) What is $E[\mathbf{T}]$ and conditional expectation $E[\mathbf{T} - t | \mathbf{T} > t]$? Based on these two expected values, what can one say in general about the “memory” of this $\mathbf{T}$ with distribution (1)? Note that a can be either positive or negative.

(c) Consider a stochastic timer follows the probability distribution given in Eq. (1). John starts the timer at time zero and records the duration and computes the average. Jane always starts s amount of time later than John starts, then records the remaining time on the same timer and computes her own average. Using the results in (b) to discuss some crucial difference for positive and negative a.