1. Consider an irreducible, four-state, continuous-time Markov chain $X(t)$ with transition rate matrix

$$Q = \{q_{ij}\}, \;\; q_{ii} = \sum_{j \neq i} q_{ij}, \;\; (i, j = 1, 2, 3, 4).$$

Let π_i be the stationary distribution of the Markov chain, that is

$$\sum_{i=1}^{4} \pi_i q_{ij} = 0, \; \forall j.$$

Furthermore, we assume that the detailed balance holds true:

$$\pi_i q_{ij} = \pi_j q_{ji}, \;\; \forall i, j.$$

If the initial distribution is

$$p_i(0) = \pi_i, \quad \text{that is} \quad \Pr\{X(0) = i\} = \pi_i,$$

then the Markov process $X(t)$ is call stationary.

(1) Show that the stationary Markov process has the trajectory

$$X(0) = i_0, \; X(t_1) = i_1, \; X(t_2) = i_2, \; \cdots, \; X(t_n) = i_n,$$

and the trajectory

$$X(0) = i_n, \; X(t_1) = i_{n-1}, \; \cdots, \; X(t_{n-1}) = i_1, \; X(t_n) = i_0,$$

with equal probability. This is called *time reversible*.

(2) Show that matrix Q is similar to a symmetric matrix. That is, there exists a matrix B: BQB^{-1} is a symmetric matrix.