1. X and Y have a bivariate normal distribution

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}e^{-Q(x,y)/2}$$

in which

$$Q(x,y) = \frac{1}{1-\rho^2}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\}.$$

Show that the conditional expectation

$$E[X|Y=y] = \mu_X + \frac{\rho\sigma_X}{\sigma_Y}(y-\mu_Y),$$

and conditional variance

$$Var[X|Y=y] = \sigma_X^2(1-\rho^2).$$

2. Let's consider a Brownian motion B_t with

$$E[B_t] = 0, \quad E[B_t^2] = \sigma^2(t).$$

(a) Show that the covariance function

$$E[B_tB_{t+s}] = \sigma^2(t)$$

in which $t, s \geq 0$.

(b) Let's assume $t_1 \leq t_2 \leq t_3$, find out

$$E[B_{t_1}B_{t_2}B_{t_3}] = ?$$

3. Let $B(t)$ be a standard Brownian motion, i.e, $E[B(t)] = 0$ and $E[B^2(t)] = t$. Show that the stochastic process defined by $W(t) = cB(t/c^2)$ is a standard Brownian motion as well.

4. Evaluate $E\left[e^{\lambda B(t)}\right]$ for an arbitrary constant λ and standard Brownian motion $B(t)$.

5. Let $B(t)$ be the standard Brownian motion, i.e., $E[B(t)] = 0$ and $E[B^2(t)] = t$. Determining the covariance functions for the stochastic processes:

(a) $U(t) = e^{-t}B\left(e^{2t}\right)$, for $t \geq 0$:

$$Cov[U(t), U(s)] = E[U(t)U(s)] - E[U(t)]E[U(s)].$$

(b) $V(t) = (1-t)B\left(t/(1-t)\right)$, for $0 < t < 1$.

(c) $W(t) = tB(1/t)$, with $W(0) = 0$.