

Gambler's Ruin and Random Walks

5.1. Gambler's Ruin

One of the interesting features of stochastic processes is that the same problem can occur with several names and can be attacked using several different methods. Let us consider the transition digraph in Figure 5.1.

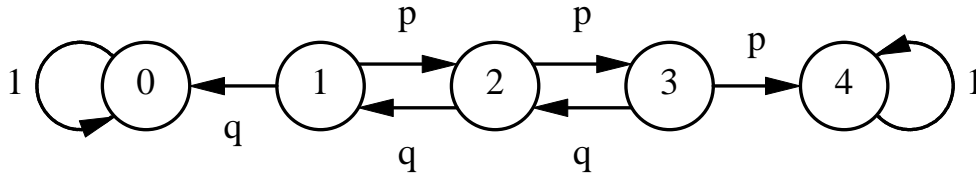


Figure 5.1 Random walk with absorbing ends

I previously referred to this sort of digraph as a drunkard's walk. It also describes a gambler's ruin. The gambler's ruin is one of the oldest and most famous problems in probability. It was originally posed by Pascal to Fermat in 1656. It was solved by Huygens and is sometimes called Huygens' fifth problem (because it was the last of a series of five problems that Huygens posed in his early treatise on probability theory). The method that we will use to solve this problem goes back to James Bernoulli (1713) and Nicholas Struyck (1716). Before outlining this solution, let me first, however, use the above digraph to describe a simple example of the problem.

Imagine two gamblers, A and B, who, between them, own \$4 and who make a series of \$1 bets against each other. The state i in this digraph is gambler A's wealth. (Please notice that I am using the state space $S = \{0, 1, 2, 3, 4\}$ instead of my more usual $S = \{1, 2, 3, 4, 5\}$.) A and B gamble against each other until either A has lost all his money and is broke or until A has won all the money and B is broke. For each bet, A has a probability p of winning and q of losing (with $p + q = 1$). What is the probability that A is ruined? How long can A expect to play?

The transition matrix for the above problem is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5.1)$$

We can, and hereby do, generalize this problem to one in which the two

players own \$N, the possible states are $S = \{0, 1, \dots, N - 1, N\}$, and we are dealing with a $(N + 1) \times (N + 1)$ transition matrix.

The above chain is a restricted random walk. It has two boundaries and an absorbing state at each boundary. We have previously determined the time to absorption and the probability that A will be ruined (will be absorbed in state 0) using the fundamental matrix. I would now like to show you an alternative approach. This alternative approach is historically important, uses the Chapman–Kolmogorov equations, and allows me to easily consider the semi-infinite problem in which A plays against an infinitely wealthy opponent.

5.2. Ruin Probability

Let us start by computing the conditional probability that gambler A, starting with i dollars, is ruined **by** time n . This is just the n -step transition probability $p_{i0}^{(n)}$. Now the Chapman–Kolmogorov equations — you remember those — tell us that

$$p_{ij}^{(n)} = \sum_{k \in S} p_{ik}^{(m)} p_{kj}^{(n-m)}, \quad 0 < m < n. \quad (5.2)$$

If we start at a transient state in our gambler’s ruin problem, we can either move one step to the right (with probability p) or one step to the left (with probability q). For $j = 0$ and $m = 1$, we thus have

$$p_{i0}^{(n)} = p p_{(i+1)0}^{(n-1)} + q p_{(i-1)0}^{(n-1)}, \quad 0 < i < N, \quad (5.3)$$

$$p_{00}^{(n)} = p_{00}^{(n-1)}, \quad p_{N0}^{(n)} = p_{N0}^{(n-1)}, \quad (5.4)$$

with the initial conditions

$$p_{i0}^{(0)} = \begin{cases} 1, & i = 0, \\ 0, & i > 0. \end{cases} \quad (5.5)$$

That is, we have conditioned over all mutually exclusive outcomes of a single bet. This is often called a *first-step analysis*.

If you stare at the above equations long enough, you should be able to convince yourself that the $p_{i0}^{(n)}$ are monotonically increasing (or nondecreasing) in n , $p_{i0}^{(n)} \geq p_{i0}^{(n-1)}$. That is the probability of ruin, starting with i dollars, forms a monotonically increasing sequence. Since this sequence is also bounded above (by one), it converges to a probability of ultimate ruin,

$$R_i \equiv \lim_{n \rightarrow \infty} p_{i0}^{(n)}, \quad (5.6)$$

starting with i dollars. Taking the limit as n goes to infinity of equations (5.3) and (5.4), we see that we can solve for the ruin probabilities by solving the difference equation

$$R_i = p R_{i+1} + q R_{i-1}, \quad 0 < i < N \quad (5.7)$$

with the boundary conditions

$$R_0 = 1, \quad R_N = 0. \quad (5.8)$$

That is, if A's initial wealth is zero, A's probability of ruin is one, and if A's initial wealth is N , A's probability of ruin is zero.

Equation (5.7) is a linear difference equation. We can solve it by trying a solution of the form $R_i = \lambda^i$. This leads us to the characteristic equation

$$p \lambda^2 - \lambda + q = 0. \quad (5.9)$$

The roots of this characteristic equation are the eigenvalues

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4pq}}{2p}. \quad (5.10)$$

This expression for the eigenvalues can be simplified by noticing that

$$(p + q)^2 = 1 \quad (5.11)$$

$$1 = p^2 + 2pq + q^2 \quad (5.12)$$

$$1 - 4pq = p^2 - 2pq + q^2 \quad (5.13)$$

$$= (p - q)^2$$

so that

$$\sqrt{1 - 4pq} = p - q. \quad (5.14)$$

Thus, for $p \neq q$,

$$\lambda_{1,2} = \frac{(p+q) \pm (p-q)}{2p} = 1, \frac{q}{p}, \quad (5.15)$$

while for $p = q$, we have a double root with

$$\lambda_{1,2} = \frac{(p+q) \pm (p-q)}{2p} = 1. \quad (5.16)$$

For $p \neq q$, the general solution to linear difference equation for the ruin probability is simply

$$R_i = c_1 + c_2 \left(\frac{q}{p} \right)^i. \quad (5.17)$$

The constants c_1 and c_2 are found by applying the boundary conditions. Thus

$$R_0 = c_1 + c_2 = 1 \quad (5.18)$$

and

$$R_N = c_1 + c_2 \left(\frac{q}{p} \right)^N = 0. \quad (5.19)$$

It follows that

$$c_1 = \frac{-(q/p)^N}{1 - (q/p)^N}, \quad c_2 = \frac{1}{1 - (q/p)^N} \quad (5.20)$$

and that our desired solution is

$$R_i = \frac{(q/p)^i - (q/p)^N}{1 - (q/p)^N}. \quad (5.21)$$

For $p = q$, the general solution, corresponding to the double root $\lambda_{1,2} = 1$, is just

$$R_i = c_1 + c_2 i. \quad (5.22)$$

The boundary conditions now take the form

$$R_0 = c_1 = 1$$

and

$$R_N = c_1 + c_2 N = 0. \quad (5.23)$$

It follows that

$$c_1 = 1, \quad c_2 = -\frac{1}{N}. \quad (5.24)$$

and that

$$R_i = 1 - \frac{i}{N} = \frac{N - i}{N}. \quad (5.25)$$

Consider, as an example, $N = 4$ and $p = q = \frac{1}{2}$. We then get

$$R_i = 1 - \frac{i}{4} \quad (5.26)$$

so that

$$R_0 = 1, \quad R_1 = 0.75, \quad R_2 = 0.5, \quad R_3 = 0.25, \quad R_4 = 0.0. \quad (5.27)$$

This agrees with our observation, from the last chapter, that powers of the transition matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (5.28)$$

go to the limit

$$\lim_{n \rightarrow \infty} \mathbf{P}^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.75 & 0 & 0 & 0 & 0.25 \\ 0.50 & 0 & 0 & 0 & 0.50 \\ 0.25 & 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5.29)$$

For $N = 4$, $p = 2/3$, and $q = 1/3$, we instead get that

$$R_i = \frac{2^{4-i} - 1}{15} \quad (5.30)$$

so that

$$R_0 = 1, \quad R_1 = \frac{7}{15}, \quad R_2 = \frac{3}{15}, \quad R_3 = \frac{1}{15}, \quad R_4 = 0.$$

It is also interesting to consider the case in which gambler A plays an infinitely wealthy opponent (see Figure 5.2.) This is an example of a restricted random walk on a semi-infinite domain. The transition matrix in this case is the semi-infinite matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ q & 0 & p & 0 & 0 & \dots \\ 0 & q & 0 & p & 0 & \dots \\ 0 & 0 & q & 0 & p & \dots \\ 0 & 0 & 0 & q & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix}. \quad (5.31)$$

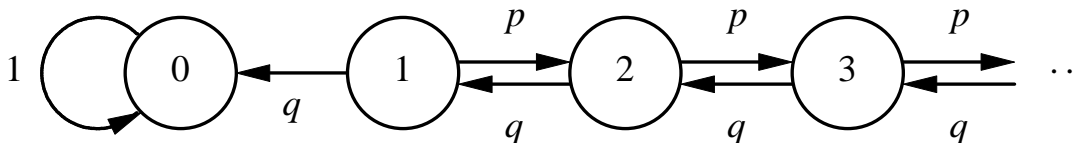


Figure 5.2 Gambler's ruin

As $N \rightarrow \infty$, we find that

$$R_i = \begin{cases} (q/p)^i, & p > q, \\ 1, & p \leq q. \end{cases} \quad (5.32)$$

That is, if A has an advantage, his (or her) probability of ruin is less than 1 and decreases exponentially with the the size of his (or her) initial captial i . Otherwise, A is certain to be ruined. You might keep this in mind the next time you decide to take on the house at one of our local Washington casinos.

5.3. Expected Duration

Another variable of interest is the expected duration of the game. The duration, of course, is the time until all the money is either gained or lost. In the last section, I wrote t_i for the expected duration, assuming that the chain starts in state i , and I will use the same notation now. Fortunately, we can condition expectations much in the same way we condition probabilities. This is sometimes known as the conditional expectation theorem and ultimately follows from the fact that that the integral or sum that defines the expectation is linear in the probability density or mass function. The conditional expectation theorem allows us to derive a difference equation for t_i using a first-step analysis.

Let us assume that A begins with i dollars. If A wins, he owns $i + 1$ dollars and the expected duration of the game is now $1 + t_{i+1}$, counting the game he just played. If A loses, he instead owns $i - 1$ dollars and the expected duration of the game is now $1 + t_{i-1}$. Conditioning on these two

possible outcomes, we see that

$$t_i = p(1 + t_{i+1}) + q(1 + t_{i-1}), \quad 0 < i < N. \quad (5.33)$$

Since $p + q = 1$, this difference equation simplifies to

$$p t_{i+1} - t_i + q t_{i-1} = -1. \quad (5.34)$$

The boundary conditions are

$$t_0 = 0, \quad t_N = 0; \quad (5.35)$$

if A has either 0 or N dollars, absorption has occurred and no more games occur.

The above difference equation is remarkably similar to the one we studied for ruin probability. The only real differences are that the equation is now nonhomogeneous and the boundary conditions are different. For $p \neq q$, the complementary solution is

$$t_i = c_1 + c_2 \left(\frac{q}{p} \right)^i. \quad (5.36)$$

A constant fails to provide a particular solution, but for a linear function, $t_i = c_3 i$, we get

$$p c_3 (i + 1) - c_3 i + q c_3 (i - 1) = -1 \quad (5.37)$$

or

$$c_3 = \frac{1}{q - p}.$$

Thus

$$t_i = c_1 + c_2 \left(\frac{q}{p} \right)^i + \frac{i}{q - p}. \quad (5.38)$$

Applying the boundary conditions, we find that

$$t_0 = c_1 + c_2 = 0, \quad t_N = c_1 + c_2 \left(\frac{q}{p} \right)^N + \frac{N}{q - p} = 0 \quad (5.39)$$

so that

$$c_1 = - \frac{N}{(q - p) \left[1 - \left(\frac{q}{p} \right)^N \right]} \quad (5.40)$$

and

$$c_2 = -c_1 . \quad (5.41)$$

After some simplification,

$$t_i = \frac{1}{q-p} \left\{ i - N \left[\frac{1 - (q/p)^i}{1 - (q/p)^N} \right] \right\} \quad (5.42)$$

For $p = q = \frac{1}{2}$, the general solution instead takes the form

$$t_i = c_1 + c_2 i - i^2 . \quad (5.43)$$

Because of the boundary conditions,

$$t_0 = c_1 = 0 , \quad t_N = c_1 + c_2 N - N^2 = 0 , \quad (5.44)$$

we have that

$$c_1 = 0 , \quad c_2 = N , \quad (5.45)$$

so that

$$t_i = i(N - i) . \quad (5.46)$$

For $N = 4$ and $p = q = \frac{1}{2}$, we get

$$t_0 = 0 , \quad t_1 = 3 , \quad t_2 = 4 , \quad t_3 = 3 , \quad t_4 = 0 . \quad (5.47)$$

This agrees with our observation, in the last chapter, that the vector of expected times to absorption, from the transient states, is

$$\mathbf{t} = \mathbf{Nc} = \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} . \quad (5.48)$$

For $N = 4$, $p = 2/3$, and $q = 1/3$, we instead get that

$$\begin{aligned} t_i &= -3 \left\{ i - 4 \left[\frac{1 - (\frac{1}{2})^i}{1 - (\frac{1}{2})^4} \right] \right\} \\ &= \frac{64}{5} [1 - (\frac{1}{2})^i] - 3i \end{aligned} \quad (5.49)$$

so that

$$t_0 = 0 , \quad t_1 = 3.4 , \quad t_2 = 3.6 , \quad t_3 = 2.2 , \quad t_4 = 0 . \quad (5.50)$$

It is interesting, once again, to consider the case in which A plays an infinitely wealthy opponent (e.g., the house). As $N \rightarrow \infty$, we find that

$$t_i = \begin{cases} \frac{i}{q-p}, & p < q, \\ \infty, & p \geq q. \end{cases} \quad (5.51)$$

So, A may actually expect to play a long time if the game is in his favor.

The gambler's ruin on a semi-infinite domain is similar to the problem that an endangered population faces. For an population, however, the odds of moving to the right or left depend upon the population size. This idea will ultimately lead to the theory of birth and death processes.

5.4. The Probability Distribution of the Duration

In the last section, we derived a formula for the expected duration of the game. One can instead ask for the entire probability distribution for the duration of the game. That is, what are the probabilities that the game will last one step, two steps, etc. ? Some day, I will add this section, but not today.

5.5. Unrestricted Random Walk

Since I've talked quite a bit about restricted random walks, I ought to say at least a few words about unrestricted random walks. Consider a simple, asymmetric, random walk in which a particle moves one step to right with probability p and one step to the left with probability q (see Figure 5.3).

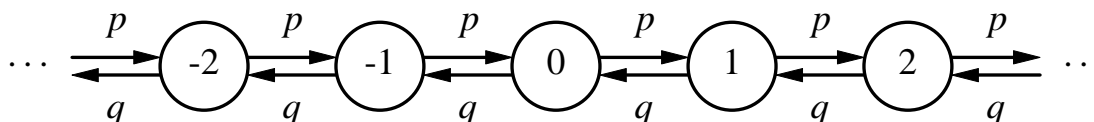


Figure 5.3 Asymmetric random walk

If $p > 0$ and $q > 0$, the chain is irreducible and periodic with period 2. One can show that if $p = q = \frac{1}{2}$, the origin (and every other state), is recurrent. You are thus certain to revisit the origin. However, one can also show, for $p = q = \frac{1}{2}$, that the mean first-return time for each state is infinite, so that each state is null-recurrent. In effect, you are certain to return, but it may take you some time to do so, especially if your decide to wander off to

infinity first. If $p \neq q$, every state is a transient state; there is some chance that you will not return to a given state.

The above conclusions change with the dimension of the walk. For a simple, symmetric walk in three dimensions, all states are transient, rather than recurrent. The probability of returning back to the origin is only 0.3405. A demonstration of these facts is, however, outside the scope of this course. They might, however, make a nice topic for a term paper.

For a simple, one-dimensional, asymmetric, random walk, it is useful and interesting to derive the n -step transition probability $p_{ij}^{(n)}$, the probability that the n th step takes the system to state j given that the system starts in state i . Determining this n -step transition probability is a first step towards proving many of the statements that I have made above.

Let me first of all emphasize that you can't always go from i to j in n steps. If i and j are an even number of steps apart, you can't go from i to j in an odd number of steps. Likewise if i and j are an odd number of steps apart, you can't go from i to j in an even number of steps. So, in everything I say below, I will be assuming the $j - i$ and n are either both even or both odd.

To reach state j at time n , the only permissible initial states are the states i that satisfy

$$j - n \leq i \leq j + n, \quad (5.52)$$

i.e., the state that are within n steps of j . To move from i to j , a particle can take k positive and $n - k$ negative steps, where k must satisfy

$$k - (n - k) = j - i. \quad (5.53)$$

Thus

$$k = \frac{1}{2} [n + (j - i)], \quad n - k = \frac{1}{2} [n - (j - i)]. \quad (5.54)$$

The probability of following any one such path is $p^k q^{n-k}$ and the probability of moving i to j in n steps is simply the sum of the probabilities over all such paths. The number of possible paths is just the number of combinations of n objects taken k at a time. Thus

$$p_{ij}^{(n)} = \binom{n}{k} p^k q^{n-k}. \quad (5.55)$$

In light of our earlier equation regarding k ,

$$p_{ij}^{(n)} = \binom{n}{\frac{1}{2}(n + j - i)} p^{\frac{1}{2}(n + j - i)} q^{\frac{1}{2}(n - j + i)}. \quad (5.56)$$

If $i = 0$, so that we start at the origin,

$$p_{0j}^{(n)} = \binom{n}{\frac{1}{2}(n+j)} p^{\frac{1}{2}(n+j)} q^{\frac{1}{2}(n-j)}. \quad (5.57)$$

I will leave it for you to show that the expected value (of the particle's position, X_n after n steps) satisfies

$$E(X_n) = n(p - q) \quad (5.58)$$

and that the variance satisfies

$$\text{Var}(X_n) = 4n p q. \quad (5.59)$$

(We will actually come back to this calculation, by another means, shortly.) The mean and the variance thus both grow linearly with time.

You should also note that

$$p_{00}^{(n)} = \binom{n}{n/2} p^{\frac{n}{2}} q^{\frac{n}{2}}. \quad (5.60)$$

Please remember that, since i and j are both zero, $j - i$ is even and n must also be even. That is, $p_{00}^{(n)}$ is zero if n is odd.

Now, you may remember that a state i is (a) recurrent (persistent) if and only if

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty \quad (5.61)$$

and is (b) transient if and only if

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty. \quad (5.62)$$

Extra Credit: For the simple, unrestricted random walk, show that the origin is recurrent if $p = q = \frac{1}{2}$ and is transient if $p \neq q$.

The above derivation of our n -step transition probability actually begs for the use of a generating (or a characteristic) function since our walk is merely the sum of n independent and identically distributed random variables. See if you remember enough about generating functions to derive this alternate proof. If you don't, don't worry: I will soon remind you all about generating functions.