Amath 568 Due Wed. 17 January, 2007

Homework Set 1

1. Using the method of dominant balance, find leading-order approximations to all roots of the following polynomials when the parameter $\varepsilon \ll 1$:
   (a) $x^2 - x - \varepsilon^{-1}$
   (b) $x^2 - \varepsilon^{-1}x - 1$
   (c) $\varepsilon x^4 + (x-1)^2$
2. Consider the ODE $y' + y\cot(x) = 0$.
   (a) Classify its singular points (including infinity).
   (b) Find the first three terms of a Frobenius series solution about $x_0 = 0$. What do we expect the radius of convergence of this Frobenius series to be?
3. Consider the quantum harmonic oscillator eigenvalue problem

$$-y'' + x^2 y = \lambda y, \quad y(x) \to 0 \text{ as } x \to \pm\infty. \qquad (*)$$

   (a) Show that the general Taylor series (TS) solution of (*) about $x_0 = 0$ has the form

$$y(x,\lambda) = a_0 \sum_{p=0}^{\infty} b_p x^{2p} + a_1 \sum_{p=0}^{\infty} c_p x^{2p+1}$$

   and find recursion relations for the coefficients b_p and c_p. What is the expected radius of convergence of this TS?
   (b) Symmetry considerations imply the eigenfunctions of (*) are all either even or odd functions of x. Using a truncated TS evaluated using Matlab or other software, find a good approximation to the even eigenfunction with the smallest eigenvalue λ_0. In particular, try to calculate λ_0 accurate to within two decimal places. Hint: Make a plot of an appropriate truncated TS $y^P(x_{large}, \lambda)$ evaluated at some suitably large x_{large} vs. λ.