AMATH 568
Differential Equations

Homework 1 Solutions

Wednesday January 17, 2006

1. Using the method of dominant balance, find leading-order approximations to all roots of the following polynomials when the parameter $\epsilon \ll 1$

   (a) $x^2 - x - \epsilon^{-1} = 0$
   Using a dominant balance of $x^2, \epsilon^{-1} \gg x$ yields

   $$\begin{aligned} x^2 - \epsilon^{-1} &\approx 0 \\ x &\approx \pm\epsilon^{-1/2} \end{aligned}$$

   Checking the assumptions: $x^2 \gg x$ gives $\epsilon^{-1} \gg \epsilon^{-1/2}$ which is indeed true for very small ϵ (the other assumption follows as well). Thus

   $$\boxed{x_{1,2} = \pm\epsilon^{-1/2}(1 + \mathcal{O}(\epsilon^{1/2}))}$$

   (b) $x^2 - \epsilon^{-1}x - 1 = 0$
   Using a dominant balance of $\epsilon^{-1}x, 1 \gg x^2$ yields

   $$\begin{aligned} -\epsilon^{-1}x - 1 &\approx 0 \\ x &\approx -\epsilon \end{aligned}$$

   Checking the assumptions: $\epsilon^{-1}x \gg x^2$ gives $1 \gg \epsilon^2$ which is indeed true for small ϵ. (the other assumption follows as well). Thus

   $$\boxed{x_1 = -\epsilon(1 + \mathcal{O}(\epsilon^2))}$$

   For the second root, assume the dominant balance $x^2, \epsilon^{-1}x \gg 1$. This yields

   $$x^2 - \epsilon^{-1}x \approx 0$$

   There are two possible solutions, however $x = 0$ does not satisfy our initial constraint. $x \approx \epsilon^{-1}$ does satisfy our constraint with an order of correction of ϵ^2. Thus

   $$\boxed{x_2 = \epsilon^{-1}(1 + \mathcal{O}(\epsilon^2))}$$

(c) $\epsilon x^4 + (x-1)^2 = 0$

Using a dominant balance of $\epsilon x^4 \ll x^2, 2x, 1$ yields

$$\begin{aligned}(x-1)^2 &\approx 0\\ x &\approx 1\end{aligned}$$

Checking the assumptions: $\epsilon \ll 1$ for small ϵ. Thus

$$\boxed{x_{1,2} = 1 + \mathcal{O}(\epsilon)}$$

Using a dominant balance of $\epsilon x^4, x^2 \gg 2x, 1$ yields

$$\begin{aligned}\epsilon x^4 + x^2 &\approx 0\\ x^2 &\approx \frac{-1}{\epsilon}\\ x &\approx \frac{\pm i}{\sqrt{\epsilon}}\end{aligned}$$

Checking the assumptions: $\epsilon(\frac{\pm i}{\sqrt{\epsilon}})^4 \gg \frac{\pm i}{\sqrt{\epsilon}}$ which is indeed true for small ϵ (the other assumption follows as well). Thus

$$\boxed{x_{3,4} = \frac{\pm i}{\sqrt{\epsilon}}(1 + \mathcal{O}(\epsilon))}$$

2. Consider the ODE $y''(x) + y(x)cot(x) = 0$

(a) Classify its singular points (including infinity).

Since $\cot(x) = \frac{\cos(x)}{\sin(x)}$, the singular points are $x = n\pi$ where $n = 0, \pm1, \pm2, \ldots$. To determine whether or not the singular points are regular or irregular, it suffices to check the following limit:

$$\lim_{x\longrightarrow n\pi} x\cot(x) = \lim_{x\longrightarrow n\pi} \frac{\cos(x) - x\sin(x)}{\cos(x)} = 1 \quad \text{by L'Hospital's Rule}$$

Thus, all of the singular points $x = n\pi$ are regular singular points.

In order to check the points at infinity, let $x = \frac{1}{t}$. The differential equation becomes

$$-t^{-2}\frac{dy}{dt} + \cot(\frac{1}{t})y = 0 \qquad \text{or} \qquad \frac{dy}{dt} - t^2\cot(\frac{1}{t}) = 0$$

The function $t \cdot (t^2\cot(\frac{1}{t}))$ is not analytic in a neighborhood of the origin (near $t = 0$, this is an essential singularity). Thus, infinity is an irregular singular point.

(b) Find the first three terms of a Frobenius series solution about $x_0 = 0$. What do we expect the radius of convergence of this Frobenius series to be?

Let $y(x) = a_0x^\alpha + a_1x^{\alpha+1} + a_2x^{\alpha+2} + a_3x^{\alpha+3} + \ldots$. Substituting this into the differential equation (and using the expansion that $\cot(x) = 1/x - \frac{1}{3}x + \mathcal{O}(x^3)$) gives:

$$\alpha a_0x^{\alpha-1} + (\alpha+1)a_1x^\alpha + (\alpha+2)a_2x^{\alpha+1} + \ldots + (\frac{1}{x} - \frac{1}{3}x + \ldots)(a_0x^\alpha + a_1x^{\alpha+1} + a_2x^{\alpha+2} + a_3x^{\alpha+3} + \ldots) = 0$$

Equating powers of x gives:

$\boxed{x^{\alpha-1}}$ $\alpha a_0 + a_0 = 0$ Since a_0 is assumed to not be zero, this yields $\alpha = -1$.

$\boxed{x^{\alpha}}$ $(\alpha+1)a_1 + a_1 = 0$. This yields $a_1 = 0$.

$\boxed{x^{\alpha+1}}$ $(\alpha+2)a_2 + a_2 - \frac{1}{3}a_0 = 0$. This yields $a_2 = \frac{a_0}{6}$

Thus

$$\boxed{y(x) = a_0 x^{-1} + 0 + \frac{a_0}{6}x + \dots}$$

where a_0 serves as a constant of integration. Since the singular point $x = 0$ is a regular singular point, we expect that the solution is valid for $x \in (-\pi, \pi)$

NOTE: The actually solution is $y(x) = c(\sin(x))^{-1}$. You can check your solution by calculating the first 3 terms in the series expansion.

3. Consider the quantum harmonic oscillator eigenvalue problem

$$-y'' + x^2 y = \lambda y \qquad y(x) \to 0 \text{ as } x \to \pm\infty$$

(a) Show that the general Taylor series (TS) solution of the quantum harmonic oscillator about $x_0 = 0$ has the form

$$y(x, \lambda) = a_0 \sum_{p=0}^{\infty} b_p x^{2p} + a_1 \sum_{p=0}^{\infty} c_p x^{2p+1}$$

and find the recursion relations for the coefficients b_p and c_p. What is the expected radius of convergence of this TS?

Let $y(x) = \sum_{j=0}^{\infty} a_j x^j$ (the TS expansion about the ordinary point $x_0 = 0$). This yields (upon re-indexing)

$$y''(x) = \sum_{j=0}^{\infty} a_{j+2}(j+1) \cdot (j+2) \cdot x^j$$

Substituting the series expansion into the differential equation gives the following for all x near zero:

$$-\sum_{j=0}^{\infty} a_{j+2}(j+1) \cdot (j+2) \cdot x^j + \sum_{j=2}^{\infty} a_{j-2} x^j = \lambda \sum_{j=0}^{\infty} a_j x^j \qquad \text{with the } x^{j+2} \text{ term reindexed}$$

We can equate powers of x as follows:

$\boxed{x^0}$ $-2a_2 = \lambda a_0$ Solving for a_2 gives

$$a_2 = -\frac{\lambda a_0}{2}$$

$\boxed{x^1}$ $-6a_3 = \lambda a_1$ Solving for a_3 gives

$$a_3 = -\frac{\lambda a_1}{6}$$

$\boxed{x^n}$ For $n \geq 2$ we get $a_{n-2} - (n+1)(n+2)a_{n+2} = \lambda a_n$ Solving for a_{n+2} gives

$$a_{n+2} = \frac{a_{n-2} - \lambda a_n}{(n+1)(n+2)}$$

Both a_0 and a_1 are arbitrary. By noticing that a_{n+2} depends only on even terms if n is even, and oddterms if n is odd, we can separate the recursive relationship into even and odd parts.
Since a_0 only enters the recursive relationship for the even terms through the term a_2, let $b_0 = 1$, $b_1 = -\frac{\lambda}{2}$. The recursive relationship for the b_p is then

$$b_0 = 1, \qquad b_1 = -\frac{\lambda}{2}, \qquad b_p = \frac{b_{p-2} - \lambda b_{p-1}}{(2p-1)(2p)} \quad \forall p \geq 1$$

Using a similar process for the odd terms, we get

$$c_0 = 1, \qquad c_1 = -\frac{\lambda}{6}, \qquad c_p = \frac{c_{p-2} - \lambda c_{p-1}}{(2p+1)(2p)} \quad \forall p \geq 1$$

Thus, the final series solution can be expressed as

$$y(x, \lambda) = a_0 \sum_{p=0}^{\infty} b_p x^{2p} + a_1 \sum_{p=0}^{\infty} c_p x^{2p+1}$$

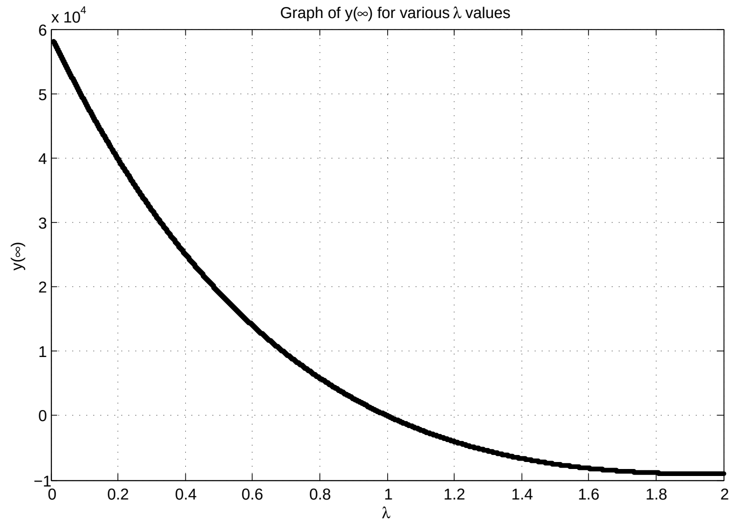
where a_0 and a_1 are arbitrary parameters, and b_p and c_p satisfy the recursive relationships

$$\begin{aligned} b_0 &= 1, \qquad b_1 = -\frac{\lambda}{2}, \qquad b_p = \frac{b_{p-2} - \lambda b_{p-1}}{(2p-1)(2p)} \quad \forall p \geq 1 \\ c_0 &= 1, \qquad c_1 = -\frac{\lambda}{6}, \qquad c_p = \frac{c_{p-2} - \lambda c_{p-1}}{(2p+1)(2p)} \quad \forall p \geq 1 \end{aligned}$$

We expect this series representation to converge for all x since the differential equation coefficient does not have any singularities in the complex plane (except at infinity).

(b) Symmetry considerations imply the eigenfunctions of the quantum harmonic oscillator are all either even or odd functions of x. Using a truncated TS evaluated using Matlab or other software, find a good approximation to the even eigenfunction with the smallest eigenvalue λ_0. In particular, try to calculate λ_0 accurate to within two decimal places.

To solve this problem, choose an x value that is large ($x_{large} \approx 5$ will work). Then, evaluate the even portion of the truncated Taylor series at this x value for many λ values and graph $y(\infty)$ vs. λ as seen here.



The boundary conditions impose that $y(\pm\infty) = 0$, so look for the λ value that forces $y(x_{large}) = 0$. Zooming in on the graph, we find that the smallest λ value is around $\lambda_0 = 1.00$.

Finally, graphing the even portion of the truncated TS with $\lambda = 1$ give the following estimate of the even eigenfunction: