Amath 568 Due Wed. 24 January, 2007

Homework Set 2

1. Using the Frobenius series for the Bessel function $J_\nu(x)$ of order ν, verify the following formulas that are sometimes useful for manipulating Bessel functions:

   (a) $J_0'(x) = -J_1(x)$

   (b) $J_{\nu-1}(x) + J_{\nu+1}(x) = 2\nu J_\nu(x)/x$

2. Consider Bessel's equation with order $\nu = 1$. In class, we calculated a Frobenius series for one solution $J_\nu(x)$, the Bessel function of the first kind. What is its leading asymptotic behavior as $x \to 0$? Because the two roots of the indicial equation are distinct but differ by an integer, the second Frobenius solution is messy and not worth finding. However, its asymptotic behavior as $x \to 0$ is a simple power law $O(x^\alpha)$. What is α?

   The Bessel function of the second kind $Y_1(x)$ is linearly independent of $J_1(x)$. Knowing nothing more about $Y_1(x)$, what does the above argument say about the asymptotic behavior of $Y_1(x)$ as $x \to 0$?

3. Consider the ODE $xy'' + (1+x)y' = 0$.

   (a) What are its two linearly independent Frobenius series solutions $y_1(x)$ and $y_2(x)$? (The problem has been chosen to minimize messy algebra, so expect some simplification).

   (b) Using a dominant balance analysis taking $y(x) = e^{S(x)}$, find the leading asymptotic behavior (i.e. keeping the first two terms S_0 and S_1) of two linearly-independent solutions $y_a(x)$ and $y_b(x)$ of this equation as $x \to \infty$. Label these solutions such that $y_b(x)/y_a(x) \to 0$ as $x \to \infty$. Again, one of the solutions has a very simple form.

   (c) Can you express the two solutions found in part (b) as linear combinations of those found in part (a), matching the behavior at large x to that at small x? This will be trivial for $y_a(x)$ but not for $y_b(x)$. You will need to find constants c_1 and c_2 such that $y_b(x) = c_1y_1(x) + c_2y_2(x)$. To do this, consider two moderately large values of x, e. g. $x_1 = 5$ and $x_2 = 6$, and determine c_1 and c_2 by simultaneously solving

   $$y_b(x_1) = c_1y_1(x_1) + c_2y_2(x_1)$$
   $$y_b(x_2) = c_1y_1(x_2) + c_2y_2(x_2)$$

   Make sure to keep enough terms in the Frobenius series to ensure they are accurate at x_1 and x_2.

   (d) Extra credit: A nice finishing touch would be to plot this matched approximate solution over $0 < x < 10$ using the Frobenius series $c_1y_1(x) + c_2y_2(x)$ for $0 < x < 5$ and the large-x asymptotic approximation $y_b(x)$ for $5 < x < 10$. They should merge smoothly into each other.