Amath 568 Due Wed. 31 January, 2007

Homework Set 3

1. Find the terms through $O(\varepsilon^2)$ in perturbation series for the three roots of $r^3 - r + \varepsilon = 0$, $\varepsilon \ll 1$.
2. Consider the ODE $y'' + 2xy' = 0$:
   (a) Using dominant balance, seek an asymptotic series for the exponentially decaying solution of this ODE as $x \to \infty$ in the form $y(x) = c\,\exp(S_0 + S_1 + S_2 \ldots)$. By matching coefficients of powers of x, show that $S_n'(x) = a_n x^{1-2n}$, where $a_0 = -2$ and
   $$a_n = \frac{1}{2}\sum_{k=0}^{n-1} a_k a_{n-k} + \left(\frac{3}{2} - n\right) a_{n-1}, \quad n \geq 1.$$
   Use this recursion to calculate a_1, a_2 and a_3.

   (b) When appropriately normalized ($c = \pi^{-1/2}$), the exponentially decaying solution of this ODE as $x \to \infty$ is $y(x) = \mathrm{erfc}(x)$. In class, we derived an asymptotic series for $\mathrm{erfc}(x)$ by integrating its definition by parts:
   $$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-y^2)\,dy = \frac{2}{\sqrt{\pi}}\exp(-x^2)\left[\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1\cdot 3}{2^3 x^5} - \frac{1\cdot 3\cdot 5}{2^4 x^7}\cdots\right]$$
   Show that the series derived in (a) is identical to this, at least through $O(x^{-5})$. In order to do this you will need to convert exp(a power series) into another power series.
3. Consider Airy's equation $y'' = xy$.
   (a) Taking $y(x) = e^{S(x)}$, show that the leading asymptotic behavior (i.e. keeping the first two terms S_0 and S_1) as $x \to \infty$ of two linearly-independent solutions of Airy's equation is
   $$y_a(x) \sim x^{-1/4}\exp(-2x^{3/2}/3)\,,\quad y_b(x) \sim x^{-1/4}\exp(2x^{3/2}/3)$$
   (b) Repeat the analysis of (a) to obtain two asymptotic solutions as $x \to -\infty$ in the form
   $$y_c(x) \sim (-x)^{-1/4}\exp(2i(-x)^{-3/2}/3) \text{ and } y_d(x) \sim (-x)^{-1/4}\exp(-2i(-x)^{-3/2}/3)$$
   We will make use of these asymptotic behaviors when we discuss WKB theory.