Amath 568 Due Wed. 7 February, 2007

**Homework Set 4**

1. Consider a mass $m$ swinging on the end of a frictionless pendulum whose length $r(\tau) = r_0\{1 + \varepsilon \sin(\omega\tau + \phi_0)\}$ is sinusoidally varied as a function of time $\tau$ with frequency $\omega$ and initial phase $\phi_0$. Assume that the counterclockwise angle $\theta(\tau)$ that the pendulum makes from the vertical is small.

(a) By equating the rate of change of angular momentum $mr^2\theta'(\tau)$ to the gravitational torque $-mgr\sin\theta \approx -mgr\theta$ on the pendulum, derive the ODE

$$r\theta'' + 2r'\theta' + g\theta = 0.$$

By defining $\phi(\tau) = r\theta/r_0$ and substituting into this ODE, show that

$$\phi'' + \frac{g - r''}{r}\phi = 0. \qquad \text{(P1)}$$

This is just like the conventional linear pendulum equation except $r$ is time-varying and gravity is reduced by the downward acceleration from the lengthening of the pendulum.

(b) Assume the oscillations of the pendulum length have small amplitude $\varepsilon \ll 1$. Show that if $\omega_0 = (g/r_0)^{1/2}$ is the natural pendulum frequency, then

$$\frac{g - r''}{r} \approx \omega_0^2 + \varepsilon(\omega^2 - \omega_0^2)\sin(\omega\tau + \phi_0).$$

Now nondimensionalize (P1) by introducing a nondimensional time $t = \omega_0\tau$. We define $\sigma = \omega/\omega_0$ to be the ratio of the length oscillation frequency to the natural pendulum frequency, and we define $y(t) = \phi(\tau)$. Since $y$ is approximately equal to $\theta$, it can be interpreted as a 'pseudoangle'. Show that

$$y'' + \{1 + \varepsilon(\sigma^2 - 1)\sin(\sigma t + \phi_0)\}y = 0. \qquad \text{(P2)}$$

(c) Now use regular perturbation theory to approximately solve (P2) subject to the initial condition $y(0) = 1$, $y'(0) = 0$, correct through $O(\varepsilon)$ terms. Show that if $\sigma = 2$, the perturbations grow proportional to $\varepsilon t$, resulting in an increase in the swinging amplitude of the pendulum. This phenomenon is called *parametric resonance*.

(d) Parametric resonance is how a child 'pumps' a swing to large amplitude. At the beginning of the swing, her legs are partially bent back. As she swings down, she swings her legs forward, temporarily moving her CG down and effectively lengthening the pendulum. While in the upswing, she straightens her legs, moving her CG up and shortening the pendulum. As she starts to swing down and backward, she bends her legs (lengthening the pendulum), but during the upswing her legs are tightly bent (shortening the pendulum), ending the cycle. Thus the pendulum is longest during the downswings and longest during the upswings, oscillating in length twice per natural period of the swing ($\sigma = \omega/\omega_0 = 2$, the resonance condition). Discuss this process by considering how much the swinging amplitude is changed in a single period (after $t = 2\pi$) as a function of the initial phase $\phi_0$ of the pendulum length. What $\phi_0$ is optimal for pumping the swing, and how does this compare to the child's strategy?