Amath 568 Due Wed. 21 February, 2007

Homework Set 5

Consider a horizontally unbounded layer of fluid at rest in a tank of depth *H*. The fluid density $\rho(z)$ varies with height *z*. Let *x* be a horizontal coordinate. If the layer is slightly perturbed to make fluid motions in the *x*-z plane of a broad horizontal scale much greater than the tank depth, the vertical velocity field $w(x, z, t)$ obeys the following PDE:

$$w_{zztt} + N^2(z)w_{xx} = 0, \text{ all } x,\ 0 < z < H, \quad w(x, 0, t) = w(x, H, t) = 0.$$

where $N(z) = \{-(g/\rho)(d\rho/dz)\}^{1/2}$ is called the buoyancy frequency. The vertical restoring force of buoyancy supports linear wave modes of the form $w(x, z, t) = \mathrm{Re}\{W(z)\exp(ik[x-ct]\}$, where $k>0$ is a specified horizontal wavenumber, and the unknown wave speed *c* must be solved for.

1. Show that $W(z)$ obeys the boundary-value problem (BVP)

$$W'' + m^2(z)W = 0, \quad 0 < z < H, \quad W(0) = W(H) = 0.$$

   where $m(z) = N(z)/c$.

2. Show that there is a WKB approximate solution to this BVP if $c = c_n = \int_0^H N(z)dz / n\pi$, n = 1, 2, 3… (called the equivalent shallow-water wave speed). Will this c_n and the corresponding WKB solution be most accurate for small *n* or for large *n*?

3. The typical density profile of the ocean can be roughly modeled by taking H = 5 km, *z* to be height above the ocean floor, and $N(z) = N_0 \exp[(z-H)/h]$, where h = 1 km and $N_0 = 0.005\ s^{-1}$. Calculate the WKB approximations to the largest three wave speeds and plot the corresponding WKB solutions $W_n(z)$. In the posted solutions, we will compare these with an 'exact' numerical solution.