Amath 568 Due Fri. 9 March, 2007

Homework Set 6

1. Determine matched inner and outer solutions of the following singularly perturbed boundary value problem:

$$\varepsilon y'' + y' + xy^2 = 0, \quad y(0) = y(1) = 1, \quad 0 < \varepsilon << 1.$$

   Plot your matched solutions for ε = 0.1 and ε = 0.01. In each case, label the boundary layer.

2. Let's recast the swing-pumping problem of Homework 4 as an autonomous nonlinear system. This problem involved a pendulum whose length $r(t)$ is deliberately varied by the swinger as a function of the pendulum angle $\theta(t)$ to the vertical. As discussed in HW4, the pendulum obeys the ODE

$$r\theta'' + 2r'\theta' + g\theta = 0. \quad (*)$$

   In solving this problem, we found that to efficiently pump energy into the swinging motion, r should be longest in the middle of the downswing (when $\theta\theta' < 0$) and shortest in the middle of the upswing ($\theta\theta' > 0$). If length and time are nondimensionalized such that the mean pendulum length and period are both 1 (this will also make the nondimensional gravity $g = 1$ in (*)), a parametric form with this property is:

$$r = 1 - \varepsilon\theta\theta'/(\theta^2 + \theta'^2), \quad 0 < \varepsilon <<1. \quad (*')$$

   (a) Using the chain rule to express r' in terms of θ and u, show that (*) can be written as the following autonomous pair of ODEs for $\theta(t)$ and $u(t) = \theta'(t)$:

$$\theta' = u\,,$$

$$(r + 2u\frac{\partial r}{\partial u})u' = -(2\frac{\partial r}{\partial \theta}u^2 + \theta)\,,$$

   where

$$r = 1 - \varepsilon\frac{\theta u}{\theta^2 + u^2}\,, \quad \frac{\partial r}{\partial \theta} = \varepsilon\frac{u(\theta^2 - u^2)}{(\theta^2 + u^2)^2}\,, \quad \frac{\partial r}{\partial u} = -\varepsilon\frac{\theta(\theta^2 - u^2)}{(\theta^2 + u^2)^2}\,.$$

   (b) Using Matlab (e.g. modifying the van der Pol example on the class web page), draw a phase portrait of this system for ε = 0.1 over the region $-1 < \theta\,, u < 1$ showing three orbits starting at $u(0) = 0$, $\theta(0) = 0.3, 0.6, 0.9$ over $0 < t < 2\pi$, plotted as a set of points in the phase plane at times that are multiples of 0.1π. We are restricting the region because the derivation of (*) was only valid for small values of θ (it would be easy to generalize the derivation to large θ, but we won't do this here).

   (c) Construct a Poincare map for this system by considering the intersection of successive orbits with the ray $u = 0$, $\theta > 0$ (again using the posted Matlab example script). Using the Poincare map, argue that the point $u = \theta = 0$ should be regarded as an unstable fixed point or 'spiral'. By the way, this fixed point is not amenable to classical linear stability analysis since the right-hand side of the ODE system is not a continuously differentiable function of u and θ at the fixed point.