# AMATH 568
# Differential Equations

Homework 6 Solutions

Friday March 9, 2007

1. Determine matched inner and outer solutions of the following singularly perturbed boundary value problem:
$$\epsilon y'' + y' + xy^2 = 0, \quad y(0) = y(1) = 1, \quad 0 < \epsilon \ll 1$$
Plot your matched solutions for $\epsilon = 0.1$ and $\epsilon = 0.01$. In each case, label the boundary layer.

**Solution:**

Assuming that the boundary layer occurs on the left hand side of the interval, let's seek an inner solution near the point $x = 0$ by letting $x = \delta X$. Recalculating the ODE, we get

$$\frac{\epsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' + \delta X Y^2 = 0, quad Y(0) = 1$$

By seeking a dominant balance solution, we can easily say that $\delta = \epsilon$. Now we seek to solve $Y'' + Y' = 0$ where $Y(0) = 1$. This yields the solution that $Y(X) = c_1 + c_2 e^{-X}$. Applying the boundary condition, we see that

$$Y(0) = c_1 + c_2 = 1 \rightarrow c_1 = 1 - c_2$$

$$Y(x) = 1 + c(e^{-X} - 1) \rightarrow y_{in}(x) = 1 + c(e^{-X/\epsilon} - 1)$$

Now, let's look for the outer solution (the solution to the unperturbed problem). For $x \gg \epsilon$, we have

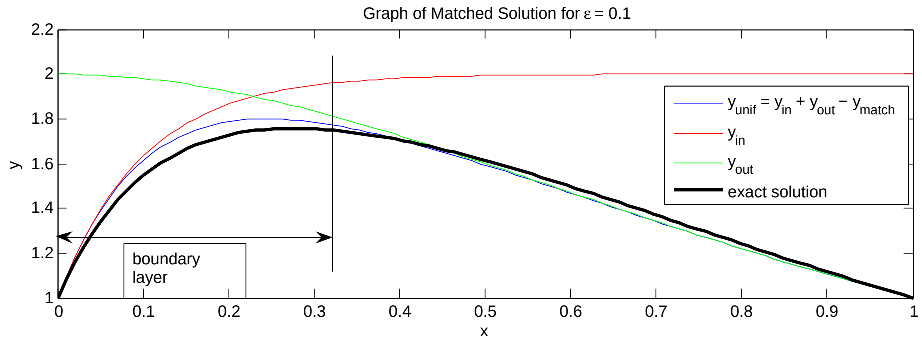
$$y' + xy^2 = 0 \quad y(1) = 1$$

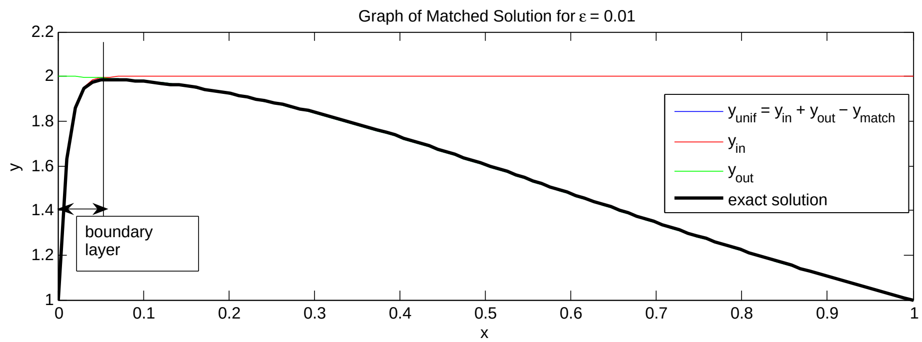
Using separation of variables, we get
$$y_{out}(x) = \frac{2}{x^2 + 1}$$

In order to determine our constant $c$ in the inner solution, we match the inner and outer solutions by comparing $\lim_{x\to\infty} y_{in} = 1 - c$ with $\lim_{x\to 0} y_{out} = 2$. Quickly, we see that $c = -1$, and $y_{in}(x) = 2 + e^{-x/\epsilon}$.

The final solution given by $y(x) = y_{in} + y_{out} - y_{match}$ is

$$\boxed{y(x) = e^{-x/\epsilon} + \frac{2}{x^2 + 1}}$$




2. Let's recast the swing-pumping problem of Homework 4 as an autonomous nonlinear system. This problem involved a pendulum whose length $r(t)$ is deliberately varied by the swinger as a function of the pendulum angle $\theta(t)$ to the vertical. As discussed in HW4, the pendulum obeys the ODE

$$r\theta'' + 2r'\theta' + g\theta = 0$$

(a) Using the chain rule to express $r'$ in terms of $\theta$ and $u$, show that (*) can be written as the following autonomous pair of ODEs for $\theta(t)$ and $u(t) = \theta'(t)$:

$$\theta\prime = u, \qquad (r + 2ur_u)u' = -(2r_\theta u^2 + \theta)$$

where

$$r = 1 - \epsilon\frac{\theta u}{\theta^2 + u^2}, \quad r_\theta = \epsilon\frac{u(\theta^2 - u^2)}{(\theta^2 + u^2)^2}, \quad r_u = \epsilon\frac{\theta(u^2 - \theta^2)}{(\theta^2 + u^2)^2}$$

**Solution:**

By letting $u = \theta'$, and by using the chain rule, we get

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} + \frac{dr}{du}\frac{du}{dt} = r_\theta u + r_u u\prime$$

where $r_\theta$ and $r_u$ depend only on the parameters $\theta$ and $u$

$$ru' + 2(r_\theta u + r_u u')u + g\theta = 0$$

Collecting $u'$ terms, we get

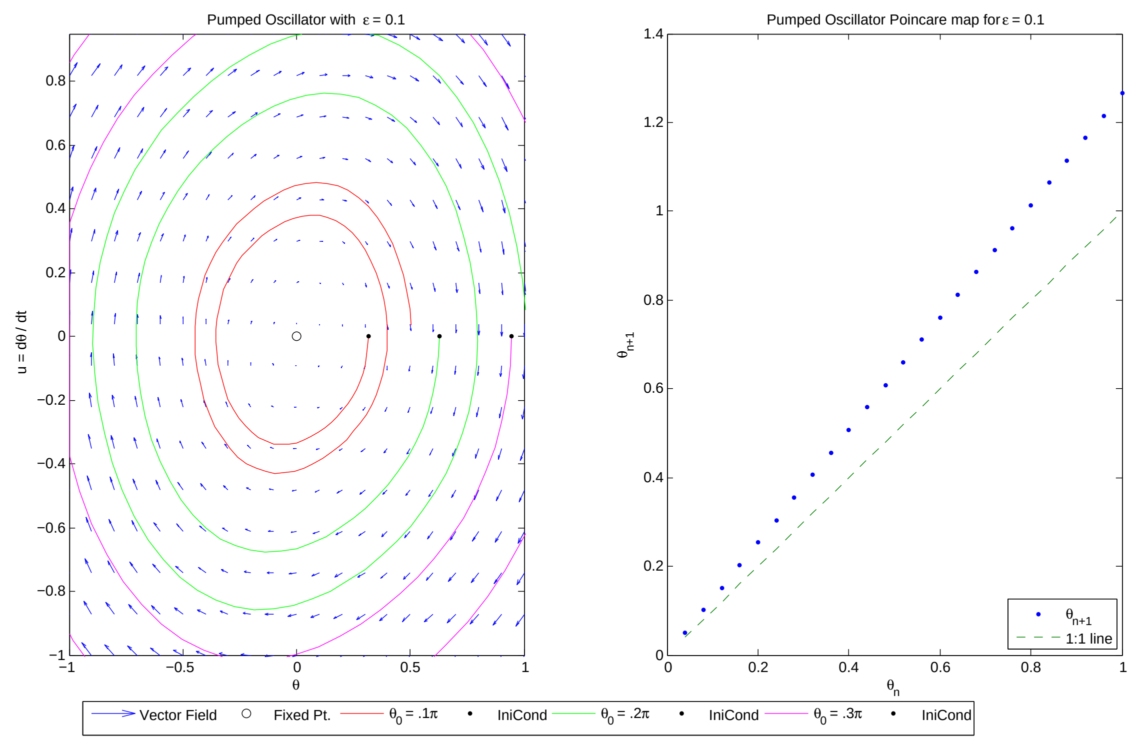
$$\boxed{(r + 2r_u u)u' = -(g\theta + 2r_\theta u^2)}$$

(b) Using Matlab (e.g. modifying the van der Pol example on the class web page), draw a phase portrait of this system for $\epsilon = 0.1$ over the region $-1 < \theta, u < 1$ showing three orbits starting at $u(0) = 0, (0) = 0.3, 0.6, 0.9$ over $0 < t < 2\pi$, plotted as a set of points in the phase plane at times that are multiples of $0.1\pi$. We are restricting the region because the derivation of (*) was only valid for small values of $\theta$ (it would be easy to generalize the derivation to large $\theta$, but we won't do this here).

**Solution: See figure in Problem 2.b**

(c) Construct a Poincare map for this system by considering the intersection of successive orbits with the ray $u = 0, \theta > 0$ (again using the posted Matlab example script). Using the Poincare map, argue that the point $u = \theta = 0$ should be regarded as an unstable fixed point or spiral. By the way, this fixed point is not amenable to classical linear stability analysis since the right-hand side of the ODE system is not a continuously differentiable function of $u$ and $\theta$ at the fixed point.

**Solution:**




By looking at the figures for the Poincare map, and comparing to a line with slope 1, we see that all of the ordered pairs $(\theta_n, \theta_{n+1})$ lie above the line. Thus, $(0, 0)$ is the only fixed point in our small $\theta$ parameter regime.

If we connect the dots of the Poincare map, we can see that it's linear with a slope $> 1$. Thus, $\theta_{n+1} = c\theta_n$ where $c > 1$. If we write out an explicit solution for $\theta_n$ we see that $\theta_n = c^n\theta_0$. Thus the amplitude after $n$ periods is larger than the original amplitude since $c^n > c$. Thus, the fixed point $(0, 0)$ is an unstable fixed point. From the phase portrait, we see that it's an unstable spiral.