(1) What do the solutions to Bessel's eqn $y'' + x^{-1}y + k^2y = 0$ look like as $x \to 0$? as $x \to \infty$? for finite x with $k \to \infty$?

(2) Can we find approximate eigenfunctions to Schrödinger's eigenvalue equation for the harmonic oscillator

$$-y'' + x^2 y = \lambda y \quad \text{with } y(x) \to 0 \text{ as } x \to \pm\infty ?$$

(3) Find the nonlinear correction to the period of a pendulum governed by

$$\ddot{\Theta} + \sin\Theta = 0 \;, \quad \Theta(0) = \varepsilon \ll 1, \quad \dot{\Theta}(0) = 0.$$

(4) Explain "parametric resonance", i.e. pumping a swing to make yourself go higher

(5) Solve $\underbrace{\varepsilon y''}_{\text{diffusion}} + \underbrace{2y'}_{\text{transport}} + \underbrace{y^2}_{\text{chemical reaction}} = 0$, $y(0) = 0$, $y(1) = 1$, $0 < \varepsilon \ll 1$.

Taylor series around ordinary points of Homogeneous linear ODEs

Consider the n'th order homogeneous linear ODE (B+O Ch 3.1-2)

$$y^{(n)}(x) + p_{n-1}(x)\, y^{(n-1)}(x) \ldots + p_1(x) y'(x) + p_0(x) y = 0 \;, \quad (*)$$

$$y^{(k)}(x) \equiv \frac{d^k y}{dx^k}$$

Assume that the $p_k(x)$ are defined for complex as well as real arguments x.

The point $x_0 \neq \infty$ is an ordinary point of $(*)$ if all $p_k(x)$, $k = 0, .., n-1$ are analytic in a neighborhood of x_0 in complex x-plane.

(e.g. Airy's equation $y'' - xy = 0$, all finite x_0 are regular points)

Theorem
All n linearly independent solutions of $(*)$ are analytic in a nbhd of an ordinary point, and have Taylor series expansions

$$y_j(x) = \sum_{m=0}^{\infty} a_{jm} (x - x_0)^m \;, \quad j = 1, .., n$$

with a radius of convergence equal to the nearest singularity of any $p_k(x)$. (Fuchs 1866)