

Homework 1: Electromagnetic Waveguides

The propagation of electromagnetic energy in a one-dimensional optical waveguide is governed by the partial differential equation:

$$2ikU_z + U_{xx} + k^2\Delta^2n(x)U = 0, \quad (1)$$

where U is the envelope of the electromagnetic field and which has been nondimensionalized such that the unit length is the waveguide core radius $a = 10\mu\text{m}$. Here, the dimensionless wavenumber $k = 2\pi n_0 a / \lambda_0$ where $n_0 = 1.46$ is the cladding index of refraction and $\lambda_0 = 1.55\mu\text{m}$ is the free-space wavelength. The parameter $\Delta^2 = (n_{core}^2 - n_0^2) / n_0^2$ measures the difference between the peak value of index in the core $n_{core} = 1.48$ and the cladding n_0 . Note that z measures the distance travelled in the waveguide and x is the transverse dimension.

A typical solution technique for this problem is to assume a solution of the form

$$U = \sum_1^N A_n \psi_n(x) \exp\left(\frac{i}{2k} \beta_n z\right) \quad (2)$$

which is called an eigenfunction expansion solution (ψ_n =eigenfunction, β_n =eigenvalue). Plugging in this solution ansatz to Eq. (1) gives the boundary value problem:

$$\frac{d^2\psi_n}{dx^2} + [k^2\Delta^2n(x) - \beta_n] \psi_n = 0 \quad (3)$$

where we expect the solution $\psi_n(x) \rightarrow 0$ as $x \rightarrow \pm\infty$ and β_n is called the propagation constant. The function $n(x)$ gives the index of refraction profile of the fiber and is typically manufactured to enhance the performance of a given application. For ideal profiles,

$$n(x) = \begin{cases} 1 - |x|^\alpha & 0 \leq |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad (4)$$

Two cases of particular interest are for $\alpha = 2$ and $\alpha = 10$.

(a) Calculate the first five *normalized* eigenfunctions (ψ_n) and eigenvalues (β_n) for these two cases using a shooting scheme. (NOTE: normalization $\int_{-\infty}^{\infty} |\psi_n|^2 dx = 1$)

(b) Calculate the first five *normalized* eigenfunctions (ψ_n) and eigenvalues (β_n) for these two cases using a direct solve scheme. Be sure to use a forward- and backward-difference scheme to approximate the boundary conditions (HINT: let $3 + \Delta x \sqrt{\beta} \approx 3$). (NOTE: normalization $\int_{-\infty}^{\infty} |\psi_n|^2 dx = 1$)

(c) For high intensity pulses, the index of refraction depends upon the intensity of the pulse itself. The propagating modes are thus found from

$$\frac{d^2\psi}{dx^2} + [\gamma|\psi|^2 + k^2\Delta^2n(x) - \beta] \psi = 0. \quad (5)$$

Depending upon the sign of γ , the waveguide leads to focusing or defocusing of the electromagnetic field. Find the first three *normalized* modes for $\gamma = \pm 0.2$ using shooting.

(d) For the case $\alpha = 2$ and for a fixed value of the propagation constant (take, for instance, β_1), perform a convergence study of the shooting method by controlling the error tolerance, TOL, in ODE45 and ODE23:

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TOL=1e-4;
OPTIONS = odeset('RelTol',TOL,'AbsTol',TOL);
[T,Y] = ODE45('F',TSPAN,Y0,OPTIONS);
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Show that indeed the schemes are fourth order and second order respectively by running the computation across the computational domain and adjusting the tolerance. In particular, plot on a log-log scale the average step-size (x-axis) using the *diff* and *mean* command versus the tolerance (y-axis) for a large number of tolerance values. What are the slopes of these lines? Note that the local error should be $O(\Delta t^5)$ and $O(\Delta t^3)$ respectively. What are the local errors for ODE113 and ODE15s?