Midterm Exam - AMATH 584
November 5, 2003
Closed Book - 50 minutes
(one sided 8.5 by 11 notes)

1. A scientist wants to fit a model to the data (t,b): (1,3), (2,6), (3,11), (4,18), and (5,30) of the form $b(t) = x_1 + x_2 t + x_3 t^2$. The equation that she would like to solve is the overdetermined system $Ax \approx b$ where A and b are given below. In terms of the matrices generated by the attached Matlab, answer the following questions:
   (a) (+5) What is rank(A)? Explain your answer.
   (b) (+10) Give three different BASES for the range of matrix A. Two of these should be orthonormal and the third just linearly independent.
   (c) (+5) Does A have a null space? Explain.
   (d) (+5) Set up the normal equations (but do not solve or multiply out the respective product of the matrices) that would need to be solved to obtain the best fit of $b(t)$ to the data in the least squares sense.
   (e) (+5) Using the SVD information, what system would you solve to find this same least squares solution? (Set it up, but don't solve.)
   (f) (+5) What is the associated residual for your solution written as an orthogonal projection of the vector b onto the proper space?
   (g) (+5) Using the QR information, what system would you solve to find this same least squares solution?
   (h) (+5) Write Ax_{LS} as an projection of the vector b onto the proper space.
   (i) (+5) In the attached Matlab, the matrix rank1A is the best rank 1 approximation to matrix A, and the matrix rank2A is the best rank 2 approximation to matrix A in the 2-norm. What is the 2-norm of the error matrix rank2A-A, i.e., $||\text{rank}2A - A||_2$.

2. (+5) Define the relative condition number, κ, of a problem with respect to perturbations in the data.

3. (+10) Consider the problem of multiplying the $n \times n$ matrix A by the data vector x. Derive that κ, the relative condition number of this problem with respect to perturbations in the data x, is given by $\kappa = ||A|| \frac{||x||}{||b||}$ and show that this is bounded above by $\kappa(A) = ||A|| ||A^{-1}||$.

4. (+10) Explain very carefully what is meant by an algorithm being backward stable and relate the relative error produced by such an algorithm in terms of the problem conditioning and the machine epsilon.

5. (+25) For the statements below, indicate whether they are True or False. If False, give a counter example, if True, say why.
   (a) The absolute error in adding two floating point numbers on a computer is no bigger than e_{mach}, the machine epsilon.
   (b) Let x and y be $m \times 1$ vectors. Then x^*y is a number and xy^* is an $m \times m$ rank-one matrix.
   (c) Householder reflectors can be used to factor an $m \times n$ matrix A into the product QR where Q is an $m \times m$ unitary matrix and R is an $m \times n$ upper triangular matrix. Furthermore, if A has full rank, the first n columns of Q form an orthonormal basis for the range of matrix A.
   (d) There is always at least one solution, x_{LS}, to a linear least squares problem.
   (e) It costs $O(n^2/2)$ multiplications to solve $Ux = y$ if U is an $n \times n$ upper triangular nonsingular matrix.

```
A=[1 1 1;1 2 4;1 3 9;1 4 16;1 5 25]

A =

     1     1     1
     1     2     4
     1     3     9
     1     4    16
     1     5    25

[u,s,v]=svd(A)

u =

   -0.0390    0.5279    0.7781   -0.0258   -0.3371
   -0.1367    0.5890   -0.0760    0.2809    0.7414
   -0.2950    0.4575   -0.4353   -0.6882   -0.2017
   -0.5137    0.1331   -0.2996    0.6367   -0.4725
   -0.7930   -0.3839    0.3309   -0.2036    0.2698


s =

   32.1563         0         0
         0    2.1977         0
         0         0    0.3744
         0         0         0
         0         0         0


v =

   -0.0553    0.6023    0.7964
   -0.2244    0.7697   -0.5977
   -0.9729   -0.2118    0.0926

[q,r]=qr(A)

q =

   -0.4472   -0.6325    0.5345   -0.0258   -0.3371
   -0.4472   -0.3162   -0.2673    0.2809    0.7414
   -0.4472    0.0000   -0.5345   -0.6882   -0.2017
   -0.4472    0.3162   -0.2673    0.6367   -0.4725
   -0.4472    0.6325    0.5345   -0.2036    0.2698
```

```
r =

   -2.2361   -6.7082  -24.5967
         0    3.1623   18.9737
         0         0    3.7417
         0         0         0
         0         0         0

b=[3 6 11 18 30]'

b =

     3
     6
    11
    18
    30

u3=u(:,1:3)

u3 =

   -0.0390    0.5279    0.7781
   -0.1367    0.5890   -0.0760
   -0.2950    0.4575   -0.4353
   -0.5137    0.1331   -0.2996
   -0.7930   -0.3839    0.3309

v3=v(:,1:3)

v3 =

   -0.0553    0.6023    0.7964
   -0.2244    0.7697   -0.5977
   -0.9729   -0.2118    0.0926

s3=s(1:3,1:3)

s3 =

   32.1563         0         0
         0    2.1977         0
         0         0    0.3744
```

```
u45=u(:,4:5)

u45 =

   -0.0258   -0.3371
    0.2809    0.7414
   -0.6882   -0.2017
    0.6367   -0.4725
   -0.2036    0.2698


q3=q(:,1:3)

q3 =

   -0.4472   -0.6325    0.5345
   -0.4472   -0.3162   -0.2673
   -0.4472    0.0000   -0.5345
   -0.4472    0.3162   -0.2673
   -0.4472    0.6325    0.5345

q45=q(:,4:5)

q45 =

   -0.0258   -0.3371
    0.2809    0.7414
   -0.6882   -0.2017
    0.6367   -0.4725
   -0.2036    0.2698


r3=r(1:3,1:3)

r3 =

   -2.2361   -6.7082  -24.5967
         0    3.1623   18.9737
         0         0    3.7417
```

```
u1=u(:,1)

u1 =

   -0.0390
   -0.1367
   -0.2950
   -0.5137
   -0.7930

v1=v(:,1)

v1 =

   -0.0553
   -0.2244
   -0.9729

rank1A=s(1,1)*u1*v1'

rank1A =

    0.0692    0.2811    1.2187
    0.2430    0.9866    4.2768
    0.5243    2.1288    9.2280
    0.9131    3.7077   16.0724
    1.4095    5.7234   24.8099

rank2A=rank1A+s(2,2)*u(:,2)*v(:,2)'

rank2A =

    0.7680    1.1741    0.9730
    1.0227    1.9830    4.0026
    1.1298    2.9026    9.0151
    1.0893    3.9330   16.0104
    0.9014    5.0740   24.9885
```