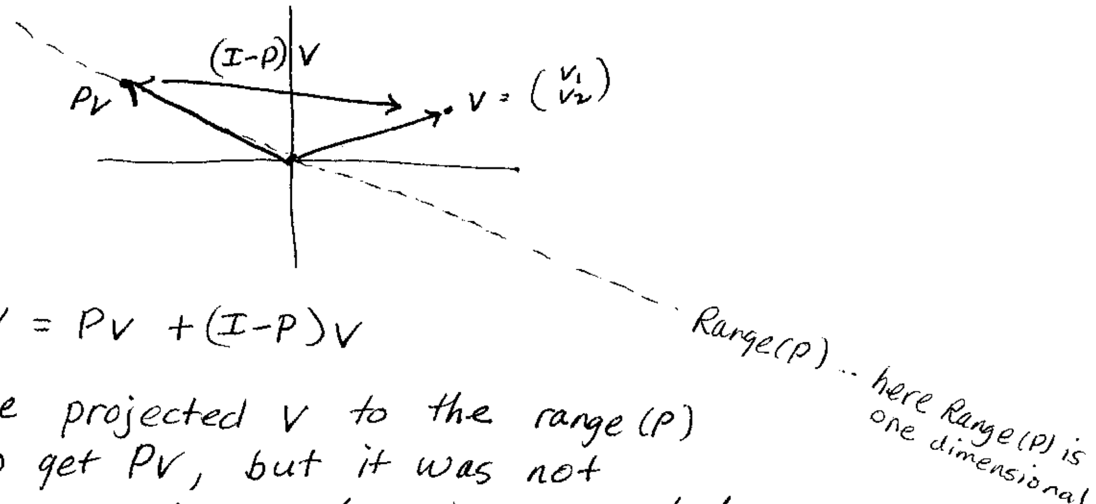
Lesson 6 : Projectors

Defn: A square matrix $P_{m\times m}$ is a projector if $P^2 = P$. (P is idempotent)

$P_{n\times n}$ projects a vector $V_{n\times 1}$ onto the range of P. (This projection may not be orthogonal.)



note: $V = PV + (I-P)V$

note: We projected V to the range (P) to get PV, but it was not done orthogonally. We projected along the $(I-P)V$ direction.

Note: $P(PV) = PV$ (PV already in range (P))

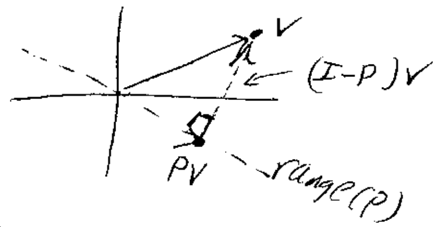
$P(I-P)V = 0$ (draw the vector $(I-P)V$ from the origin and projection along $(I-P)V$ and of couse get back to origin!)

$(P-P^2)V = 0$

So, projector P and $I-P$ (also a projector) completely divide V into 2 vectors.

$$V = \underbrace{PV}_{\in \text{range}(P)} + \underbrace{(I-P)V}_{\in \text{null}(P)}$$

Orthogonal Projectors: If Pv and $(I-P)v$ are orthogonal, then the projector P is called a orthogonal projector.



Thm: A projector P is orthogonal iff $P = P^*$.

Pf: Assume $P = P^*$ and show Pv and $(I-P)v$ are orthogonal.

$(Pv)^*(I-P)v = v^* P^*(I-P)v = v^*(P^* - P^*P)v$.

Now ~~since~~ $P^* = P$, we get $v^*(P^* - P^*P)v = v^*(P - P^2)v = 0$ since P is a projector.

Now assume P is an orthogonal projector and prove that $P = P^*$. Let $\{q_1, q_2, \dots, q_n\}$ be a basis for range(P) and $\{q_{n+1}, \dots, q_m\}$ be a basis for range$(I-P)$. Since P is a projector, we know $Pq_1 = q_1$, $Pq_2 = q_2$, $\dots$ $Pq_n = q_n$ and $Pq_{n+1} = 0, \dots, Pq_m = 0$.

Put these statement in matrix form as

$$P\underbrace{[q_1\, q_2\, q_n \mid q_{n+1} \cdots q_m]}_{Q \text{ is unitary}} = [q_1\, q_2 \cdots q_n, 0, 0, \dots 0]$$

Multiply both sides by Q^* to get

$$Q^*_{m\times m} P_{m\times m} Q_{m\times m} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}_{m\times m} = \Sigma \quad \Rightarrow \quad P = Q\Sigma Q^* \text{ and}$$

$$P^* = Q\underset{\substack{\uparrow \\ \Sigma^* = \Sigma}}{\Sigma^*} Q^* = P$$

QED

Example: Let q be an $m\times 1$ vector.

When is $P = qq^*$ a projector?

When is P an orthogonal projector?

Since P is symmetric (q real) or Hermitian it will be an orthogonal projector whenever $P = P^2$.

$$P^2 = q\underbrace{q^* q}_{\#} q^* = (q^*q)\,qq^*$$

$P^2 = P$ only when $q^*q = 1$. We can therefore conclude $\tilde{P} = \dfrac{qq^*}{q^*q}$ is always an orthogonal projector.

Example: $I - \tilde{P}$ is also an orthogonal projector

Example: Let $Q_{m\times n} = [q_1\ q_2\ \dots\ q_n]$ be n colmns of $m\times 1$ vectors that are orthonormal. Then $P = QQ^*$ is an orthogonal projector.

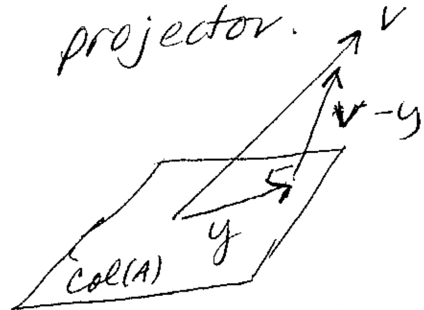
$$\left(P^2 = Q\underbrace{Q^*Q}_{I}Q^* = P\right).$$

Note: Orthogonal projectors $\neq$ orthogonal matrices unless they are also nonsingular. Usually they are rank (n) where $n<m$.

Sometimes we do not have an <u>orthonormal</u> basis $(q_1\ q_2\ \dots\ q_n)$ for the space we want to project onto. Suppose $(a_1, a_2, \dots, a_n)$ are n linearly independent vectors that make up the columns of matrix $A_{m\times n}$. (Each a_i is $m \times 1$).

<u>Projector (Orthogonal) ONTO subspace spanned by $\{a_1\ a_2 \dots a_n\}$</u>
(Project Orthogonally Onto Column Space of A)

Let's <u>derive</u> the formula for this projector.



$v = y + v - y$

y is the <u>projection</u>.

Note that $v-y$ is $\perp$ to every vector $a_1, a_2, \dots, a_n$. This says

$$A^*(v-y) = \begin{pmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{pmatrix}(v-y) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underline{0}$$

This says $A^* v = A^* y$. (1)

We want to solve for y which will reveal the projector matrix. We know that $y = Ax$ for some vector x since $y \in col(A)$. Plug this into (1) to get $A^*v = A^*Ax$. Since A has linearly independent colms (n of them), the $n\times n$ matrix A^*A is nonsingular

Which means we can solve for x.

$$x = (A^*A)^{-1}A^*v \qquad (2)$$

Multiply both sides of (2) by A to get

$$y = Ax = \underbrace{A(A^*A)^{-1}A^*}v$$

$\Downarrow$

must be projector P that projects v orthogonally to col(A).

$$\boxed{P = A(A^*A)^{-1}A^*}$$

Note: P is symmetric!

Note: If A does have orthonormal columns, $A^*A = I \Rightarrow P = AA^*$ the simplified form if we know the an orthonormal basis for the span we are projecting onto.

The formula $P = A(A^*A)^{-1}A^*$ projects orthogonally onto the column space of A in the case that the columns of A are linearly independent.

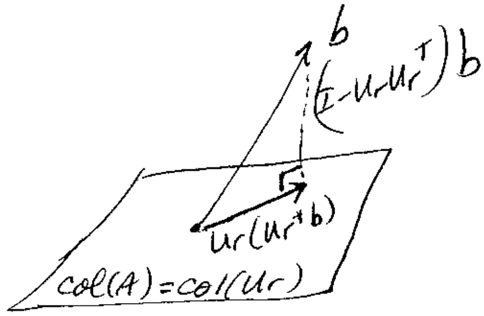
What if $A_{m \times n}$ has rank $= r$ where $r < n$? then A does not have linearly independent Columns. How do we still project a vector onto the column space of A?

Answer: Use the SVD.

$$A = U_r \Sigma_r V_r^T$$

so we know $\text{range}(A) = \text{col}(A) =$ span of the columns of U_r. We are even lucky that these columns of U_r are orthogonal and orthonormal, hence,

$$P = U_r U_r^*$$



projects b onto $\text{col}(A)$:

The projection $Pb = U_r(U_r^* b)$

$$b = \underbrace{U_r(U_r^* b)}_{\text{the projection vector}} + \underbrace{(I - \underbrace{U_r U_r^T}_{\text{also a projector.}})b}_{\text{gives the difference between } b \text{ and this projection vector}}$$